147. *Some Conditions on an Operator Implying Normality. III*

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The purpose of this note is to record some generalizations of results proved recently by I. Istrăţescu [9].

Notations. If T is an operator (bounded linear, in Hilbert space), we write $\sigma(T)$ for the spectrum of T, $\omega(T)$ for the Weyl spectrum of T, $W(T)$ for the numerical range of T and $\mathrm{Cl}\, W(T)$ for its closure, and $\hat{T}$ for the image of T in the Calkin algebra (the algebra of all operators modulo the ideal of compact operators). We refer to [2]-[4] or [7] for terminology.

Theorem 1. *If T is a seminormal operator such that $T^p = ST^{*p}S^{-1} + C$, where p is a positive integer, C is compact, and $0 \notin \mathrm{Cl}\, W(S)$, then T is normal.*

Proof. By hypothesis, $\hat{T}^p = \hat{S}\hat{T}^{*p}\hat{S}^{-1}$; moreover, it is easy to see that $\bar{W}(\hat{S}) \subset \bar{W}(S) = \mathrm{Cl}\, W(S)$, where $\bar{W}$ denotes closed numerical range [5, Theorem 3], thus $0 \notin \bar{W}(\hat{S})$. By a theorem of J. P. Williams [12], $\sigma(\hat{T}^p)$ is real, i.e., $\{\lambda^p : \lambda \in \sigma(\hat{T})\}$ is real, thus $\sigma(\hat{T})$ lies entirely on p lines through the origin. Since $\partial\omega(T) \subset \sigma(\hat{T})$, where ∂ denotes boundary (this is true for any operator [cf. 6, Theorem 2.2]), it follows that $\omega(T)$ also lies on these lines, and in particular $\omega(T)$ has zero area. Since Weyl's theorem holds for T [1, Example 6], $\sigma(T) - \omega(T)$ is countable; thus $\sigma(T)$ also has zero area, therefore T is normal by a theorem of C. R. Putnam [11].

{The following argument is of interest because it uses far less than the full force of Putnam's deep theorem. Assuming T is a seminormal operator such that $\omega(T)$ lies on finitely many lines through (say) the origin, we assert that T is normal. We can suppose T hyponormal. Writing $T = T_1 \oplus T_2$ with T_1 normal and $\sigma(T_2) \subset \omega(T)$ [3, Corollary 6.2], we are reduced to the case that $\sigma(T)$ lies on finitely many lines through the origin. Assume to the contrary that T is nonnormal. Splitting off the maximal normal direct summand of T, we can suppose that T has no normal direct summands. In particular, $\sigma(T)$ can have no isolated points (these would be eigenvalues, with reducing eigenspaces). Rotating T by a scalar of absolute value 1, we can suppose that the positive real axis contains a point of $\sigma(T)$ of maximum modulus, say

b. Then, for suitable a, $0 < a < b$, the vertical strip $\{\alpha+i\beta : a \le \alpha \le b,\ \beta \text{ real}\}$ intersects $\sigma(T)$ only at points of $[a, b]$. Let $T=H+iJ$ be the Cartesian form of T and let $H=\int \lambda\, dE$ be the spectral representation of H. Since b is not an isolated point of $\sigma(T)$, $(a, b)\cap\sigma(T)\neq\emptyset$; moreover, $\mathrm{Re}\,\sigma(T)=\sigma(H)$ [10, Theorem I], thus $(a, b)\cap\sigma(H)\neq\emptyset$ and therefore $E((a, b))\neq 0$. Thus, writing $\Delta=[a, b]$, we have also $E(\Delta)\neq 0$. Let T_Δ be the restriction of $E(\Delta)TE(\Delta)$ to the range of $E(\Delta)$ (i.e., the compression of T to that subspace). Then T_Δ is hyponormal, and $\sigma(T_\Delta)\subset\Delta$ (cf. [10, proof of Theorem II] or [11, proof of Lemma 3]); it follows that T_Δ is normal (in fact, self-adjoint [10, Corollary of Theorem I]) and is therefore a direct summand of T [11, Lemma 5], a contradiction.}

Theorem 2. *If T is an operator such that* (1) $\sigma(\hat{T})=\{0\}$, (2) *T is reduced by each of its finite-dimensional eigenspaces, and* (3) *T is reduction-spectraloid, then T is normal and compact.*

Proof. Condition (3) means that every direct summand of T is spectraloid (an operator is spectraloid if its numerical radius and spectral radius coincide). Since $\partial\omega(T)\subset\sigma(\hat{T})=\{0\}$, it follows that $\omega(T)=\{0\}$. Let $\mathcal{M}$ be the closed linear span of the finite-dimensional eigenspaces of T, and let $T_1=T\,|\,\mathcal{M}$, $T_2=T\,|\,\mathcal{M}^\perp$; thus $T=T_1\oplus T_2$, where T_1 is normal and T_2 has no eigenvalues of finite multiplicity [3, Proposition 4.1]. We assert that $T_2=0$ (therefore $T=T_1\oplus 0$ is normal). Since $\omega(T)=\omega(T_1)\cup\omega(T_2)$ [1, Example 5] and $\omega(T_2)=\sigma(T_2)$ [1, Lemma 1], we have $\sigma(T_2)=\omega(T_2)\subset\omega(T)=\{0\}$; by hypothesis, T_2 is spectraloid, therefore $T_2=0$. Thus T is normal; moreover, T is compact ([1, Example 7] or [3, remarks following Corollary 6.3]), i.e., $\hat{T}=0$.

Theorem 3. *If T is an operator such that* (1) *$\sigma(\hat{T})$ is countable,* (2) *T is reduced by each of its eigenspaces, and* (3) *T is reduction-isoloid, then T is normal.*

Proof. Condition (3) means that every direct summand of T is isoloid (an operator is isoloid if every isolated point of its spectrum is an eigenvalue). Since $\partial\omega(T)\subset\sigma(\hat{T})$, $\omega(T)$ is also countable. (Indeed, $\omega(T)=\partial\omega(T)$; if, on the contrary, $\omega(T)$ had an interior point λ, then every ray from λ would exit $\omega(T)$ at a boundary point.) Let $\mathcal{M}$ be the closed linear span of the eigenspaces of T, and let $T_1=T|\mathcal{M}$, $T_2=T|\mathcal{M}^\perp$; thus $T=T_1\oplus T_2$, where T_1 is normal and T_2 has no eigenvalues [3, Proposition 4.1]. We assert that $\mathcal{M}^\perp=\{0\}$ (therefore $T=T_1$ is normal). Assume to the contrary. As argued in the proof of Theorem 2, $\sigma(T_2)=\omega(T_2)\subset\omega(T)$, therefore $\sigma(T_2)$ is also countable (and nonempty, because $\mathcal{M}^\perp\neq\{0\}$); it follows that $\sigma(T_2)$ has at least one isolated point, and therefore, by (3), an eigenvalue, a contradiction.

Remarks. Theorem 1 is proved in [9, Theorem 1] with an added hypothesis on $\sigma(T)$.

The following remarks show that either Theorem 2 or 3 generalizes [9, Theorem 2]. (i) If $T=Q+C$, where Q is quasinilpotent and C is compact, then $\sigma(\hat{T})=\sigma(\hat{Q})\subset\sigma(Q)=\{0\}$. (ii) If T is convexoid and $\sigma(T)$ lies on a convex curve, then every eigenvalue of T lies on the boundary of $W(T)$, therefore every eigenspace of T reduces T [8, Satz 2]. (iii) Every convexoid operator is spectraloid [7, p. 115]. (iv) If T is restriction-convexoid (i.e., if the restriction of T to every invariant subspace is convexoid), then T is isoloid [2, Lemma 2], and therefore restriction-isoloid.

Theorem 4 of [9] is as follows: If T is an operator such that (1) T is polynomially compact, (2) $\sigma(T)$ lies on a convex curve, and (3) T is restriction-convexoid, then T is normal. In view of remarks (ii) and (iv) above, this theorem is extended by either of the following results: If T is (1) polynomially compact, (2′) reduced by each of its finite-dimensional eigenspaces, and (3) restriction-convexoid, then T is normal [3, Theorem 6.7]. If T is (1) polynomially compact, (2″) reduced by each of its eigenspaces and (3′) reduction-isoloid, then T is normal [3, Theorem 6.5].

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