144. *On the Index of a Semi-free S^1-action*

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1. Introduction. Let G be a compact Lie group, M^n a closed smooth n-manifold and $\varphi: G\times M^n\to M^n$ a smooth action. Then the fixed point set is a disjoint union of smooth k-manifolds F^k, $0\leqq k\leqq n$.

P. E. Conner and E. E. Floyd [2] obtained several properties of fixed point sets of smooth involutions and one of their results is the following.

Suppose that $T: M^{2k}\to M^{2k}$ is a smooth involution on a closed manifold of odd Euler characteristic. Then some component of the fixed point set is of dimension $\geqq k$.

Now we consider semi-free smooth S^1-actions on oriented manifolds and we claim the following

Theorem 1.1. *Let M^n be an oriented closed smooth n-manifold and $\varphi: S^1\times M^n\to M^n$ a semi-free smooth action. Then each k-dimensional fixed point set F^k can be canonically oriented and the index of M^n is the sum of indices of F^k, that is,*

$$I(M^n)=\sum_{k=0}^{n} I(F^k).$$

Theorem 1.2. *Suppose that $\varphi: S^1\times M^{4k}\to M^{4k}$ is a semi-free smooth S^1-action on an oriented closed manifold of non-zero index. Then some component of the fixed point set is of dimension $\geqq 2k$.*

Detailed proof will appear elsewhere.

2. Outline of the proof of Theorem 1.1.

Let S^1 and D^2 denote the unit circle and the unit disk in the field of complex numbers. Regard S^1 as a compact Lie group. Let M^n be an oriented closed smooth n-manifold and $\varphi: S^1\times M^n\to M^n$ a smooth action. The action φ is called semi-free if it is free outside the fixed point set. Then we have the following ([4], Lemma 2.2).

Lemma 2.1. *The normal bundle of each component of the fixed point set in M^n has naturally a complex structure, such that the induced S^1-action on this bundle is a scalar multiplication.*

From this lemma, a codimension of each component of the fixed point set in M^n is even. Let ν^k denote the complex normal bundle to F^{n-2k}. Then ν^k is canonically oriented and F^{n-2k} can be so oriented that the bundle map $\tau(F^{n-2k})\oplus\nu^k\to\tau(M^n)$ is orientation preserving, where $\tau(M)$ denotes the tangent bundle of M.

For each complex vector bundle ξ over an oriented closed smooth manifold X, let $S(\xi)$ and $\boldsymbol{CP}(\xi)$ denote the sphere bundle and the complex projective bundle associated to ξ, respectively. Then the orientations of $S(\xi)$ and $\boldsymbol{CP}(\xi)$ are induced by those of X and ξ. Then we shall have the following

Lemma 2.2. *Let M^n be an oriented closed smooth n-manifold, $\varphi: S^1\times M^n\to M^n$ a semi-free smooth action and F^{n-2k} an oriented $(n-2k)$-dimensional fixed point set. Let ν^k denote the complex normal bundle to F^{n-2k}. Then*

(a) $\sum_{k\geqq 1}[\boldsymbol{CP}(\nu^k)]=0$

and

(b) $[M^n]=\sum_{k\geqq 0}[\boldsymbol{CP}(\nu^k\oplus\theta^1)]$

in the oriented cobordism ring Ω_*, where θ^1 is a trivial complex line bundle.

Now we consider the index of $\boldsymbol{CP}(\xi^k)$, the total space of the complex projective bundle associated to a complex k-plane bundle ξ^k over an oriented closed manifold V^n. The following lemma is an immediate consequence of [1].

Lemma 2.3. $I(\boldsymbol{CP}(\xi^k))=\dfrac{1+(-1)^{k-1}}{2}\cdot I(V^n)$.

Combining these lemmata, we easily have

$$\sum_{k:\text{odd}} I(F^{n-2k})=0,$$
$$\sum_{k:\text{even}} I(F^{n-2k})=I(M^n)$$

Since the codimension of each component of the fixed point set is even, we have

$$I(M^n)=\sum_{k=0}^{[\frac{n}{2}]} I(F^{n-2k}) .$$

Hence we obtain Theorem 1.1.

3. Outline of the proof of Theorem 1.2.

Let

$$\Delta:\Omega_n(\boldsymbol{CP}^\infty)\to\Omega_{n-2}(\boldsymbol{CP}^\infty)$$

be the Smith homomorphism (cf. [2], § 26) and

$$i_*:\Omega_n(\boldsymbol{BU}(k))\to\Omega_n(\boldsymbol{BU}(k+1))$$

a homomorphism induced by the canonical inclusion map $i:\boldsymbol{BU}(k)\to\boldsymbol{BU}(k+1)$. Let

$$\partial:\Omega_n(\boldsymbol{BU}(k))\to\Omega_{n+2k-2}(\boldsymbol{CP}^\infty)$$

be a homomorphism as follows (cf. [4], § 3). To each complex vector bundle ξ^k we have a line bundle $\hat{\xi}$ associated to the principal S^1-bundle $S(\xi^k)\to\boldsymbol{CP}(\xi^k)$, then $\partial([\xi^k])=[\hat{\xi}]$. Then we have the following commutative diagram (cf. [2], 26.4)

$$\begin{array}{ccc} \Omega_n(\boldsymbol{BU}(k)) & \xrightarrow{\partial} & \Omega_{n+2k-2}(\boldsymbol{CP}^\infty) \\ \downarrow{i_*} & & \uparrow{\Delta} \\ \Omega_n(\boldsymbol{BU}(k+1)) & \xrightarrow{\partial} & \Omega_{n+2k}(\boldsymbol{CP}^\infty). \end{array}$$

And we obtain the following result by the same way as in the case of [2; Theorem 27.3].

Lemma 3.1. *Let $\varphi: S^1 \times M^n \to M^n$ be a semi-free smooth S^1-action on an oriented closed manifold of non-zero index, and let ν^k denote the complex normal bundle to $(n-2k)$-dimensional fixed point set F^{n-2k}. There exists a k such that $[\nu^k]$ is not in the image of $i_*: \Omega_{n-2k}(\boldsymbol{BU}(k-1)) \to \Omega_{n-2k}(\boldsymbol{BU}(k))$.*

Since $i_*: \Omega_m(\boldsymbol{BU}(k-1)) \cong \Omega_m(\boldsymbol{BU}(k))$ for $m \leqq 2(k-1)$, Theorem 1.2 is an immediate corollary of the above result.

4. Remark.

The following proposition shows that Theorem 1.2 is a generalization of the theorem of Conner and Floyd stated in the introduction of the present paper when we restrict ourselves to involutions induced from semi-free S^1-actions on oriented manifolds.

Proposition 4.1. *For an oriented closed n-manifold M^n,*

$$I(M^n) \equiv \chi(M^n) \bmod 2,$$

where $\chi(M)$ denotes the Euler characteristic number of M.

References

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