191. *Noetherian QF-3 Rings and Two-sided Quasi-Frobenius Maximal Quotient Rings**)

By Kiiti MORITA

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1970)

The notion of QF-3 rings was introduced originally by R. M. Thrall [13] for the case of finite dimensional algebras over a field. Recently this notion has been extended to the case of general rings in various ways (cf. [2], [5], [12]). For example, a ring A is called left QF-3 or left QF-3′ according as A has a faithful, projective injective left ideal or the injective envelope of the left A-module $A, E({}_AA)$, is torsionless, and a ring which is left QF-3 by any one of the definitions given in the literature is left QF-3′. The notion of QF-3′ rings, however, does not seem to fit Noetherian rings.

In this paper we shall call a ring A a left QF-3 ring if $E({}_AA)$ is flat. Right QF-3 rings are defined similarly. For example, the ring of rational integers is left QF-3 in our sense, but not left QF-3′ in the sense mentioned above. As for Noetherian QF-3 rings, we shall prove the following theorems.

Theorem 1. *Let A be a left Noetherian ring. If A is left QF-3, then A is right QF-3.*

Theorem 2. *Let A be a left Noetherian, left QF-3 ring. Then we have*

(1) $$\mathrm{Hom}_A([\mathrm{Ext}_A^n({}_AX, {}_AA)]_A, E(A_A))=0, \qquad n=1,2,\cdots$$

for every finitely generated left A-module X.

According to Jans [4], the dual of every finitely generated right A-module is reflexive if and only if

(2) $$\mathrm{Hom}_A([\mathrm{Ext}_A^1(X, {}_AA)]_A, A_A)=0$$

for every finitely generated torsionless left A-module X.[1] Hence we obtain from Theorem 2 the following

Corollary 3. *Let A be a left Noetherian left QF-3 ring. Then the dual of every finitely generated right A-module is reflexive.*

The above notion of QF-3 rings is useful also for non-Noetherian rings.

As is known as a theorem of R. E. Johnson, the maximal left

---

*) Dedicated to Professor K. Asano on his sixtieth birthday.

1) This result is true without any finiteness condition on A, although Jans assumes A to be left and right Noetherian. This fact has already been used in our previous paper [9].

quotient ring of A is semi-simple Arthinian if and only if A has zero left singular ideal and ${}_AA$ is finite dimensional. Suppose that this is the case with A. From the main theorem in R. R. Colby and E. A. Rutter, Jr. [2] it is seen that, under an assumption that both the left and the right socles of A are essential, the maximal left quotient ring of A is a maximal right quotient ring of A if and only if A has a faithful, projective, injective left ideal. Our notion of QF-3 rings makes it possible to remove the above assumption. Namely, it holds that the maximal left quotient ring of A is a maximal right quotient ring of A if and only if A is left QF-3 in our sense.[2] More generally, we can establish the following theorem.

Theorem 4. *Let A be a ring such that its maximal left quotient ring Q is an injective cogenerator in the category ${}_Q\mathfrak{M}$ of all left Q-modules. Then Q is a maximal right quotient ring of A if and only if A is left QF-3.*

In case Q is a quasi-Frobenius ring, Q is an injective cogenerator in ${}_Q\mathfrak{M}$ and hence Theorem 4 is applicable to this case. It is to be noted that quite recently K. Masaike [7] has obtained a characterization of rings with quasi-Frobenius maximal left quotient rings.

Throughout this paper, it is assumed that every ring has an identity and that every module is unitary.

1. Proof of Theorem 1. A slight modification of the proof of M. Harada [3, Theorem 1] is available for the present case. For the sake of completeness, we give it here.

Let R be the additive group of rational numbers and Z the ring of integers. Let us put

$$V_A=\mathrm{Hom}_Z(E({}_AA), R/Z). \tag{3}$$

Since R/Z is an injective cogenerator in the category of all Z-modules and $E({}_AA)$ is flat, V_A is injective and faithful. Let X be any finitely generated left A-module. Then by [1, Proposition 5.3, p. 120] there is a natural isomorphism

$$V_A\otimes_A X\cong \mathrm{Hom}_Z(\mathrm{Hom}_A({}_AX, {}_A[E({}_AA)]_Z), [R/Z]_Z).$$

Hence, if $0\to {}_AX'\to {}_AX$ is exact where ${}_AX, {}_AX'$ are finitely generated, so is the sequence $0\to V_A\otimes_A X'\to V_A\otimes_A X$. This proves the flatness of V_A.

Since V_A is faithful, A_A is isomorphic to a submodule of a direct product W_A of copies of V_A. By [1, Exercise 4, p. 122] W_A is flat. Since W_A is injective, $E(A_A)$ is isomorphic to a direct summand of W_A. Hence $E(A_A)$ is flat. This completes the proof.

2. Proof of Theorem 2. Let X be a finitely generated left A-

2) After this paper had been completed, there appeared a paper by V. C. Cateforis: Two-sided semisimple maximal quotient rings. Trans. Amer. Math. Soc., **149**, 339–349 (1970), in which essentially the same result is obtained by a different method.

module and let

(4) $$0 \longleftarrow X \longleftarrow P_0 \xleftarrow{\varphi_1} P_1 \xleftarrow{\varphi_2} P_2 \longleftarrow$$

be a resolution of X such that each P_i is finitely generated and projective. Then we have the following commutative diagram ($n \geqq 1$):

$$\begin{array}{ccccc}
\mathrm{Hom}_A(P_{n-1}, A)\otimes L & \xrightarrow{\mathrm{Hom}(\varphi_n, 1)\otimes 1} & \mathrm{Hom}_A(P_n, A)\otimes L & \xrightarrow{\mathrm{Hom}(\varphi_{n+1}, 1)\otimes 1} & \mathrm{Hom}_A(P_{n+1}, A)\otimes L \\
\downarrow \Phi(P_{n-1}) & & \downarrow \Phi(P_n) & & \downarrow \Phi(P_{n+1}) \\
\mathrm{Hom}_A(P_{n-1}, L) & \xrightarrow{\mathrm{Hom}(\varphi_n, 1)} & \mathrm{Hom}_A(P_n, L) & \xrightarrow{\mathrm{Hom}(\varphi_{n+1}, 1)} & \mathrm{Hom}_A(P_{n+1}, L)
\end{array}$$

where we put

(5) $$L = E({}_A A),$$

and $\Phi(P_i)$ is defined by $[\Phi(P_i)(\alpha\otimes y)](x) = \alpha(x)y$ for $\alpha \in \mathrm{Hom}_A(P_i, X)$, $x \in P_i, y \in L$.

Since L is injective and vertical maps are isomorphisms, the top row is exact. Since L is flat, we have

(6) $$\mathrm{Ext}^n_A(X, {}_A A)\otimes L = 0, \qquad n \geqq 1.$$

By retaining the meaning of R and Z as in the proof of Theorem 1 we put

$$V_A = \mathrm{Hom}_Z({}_A L, R/Z).$$

Then from (6) we get

$$\mathrm{Hom}_A([\mathrm{Ext}^n_A(X, {}_A A)]_A, V_A) = 0,$$

and hence

(7) $$\mathrm{Hom}_A([\mathrm{Ext}^n_A(X, {}_A A)]_A, W_A) = 0,$$

where W_A is a direct product of copies of V_A which contains A_A. It is to be noted here that V_A is faithful and injective. Since $E(A_A)$ is isomorphic to a submodule of W_A, we obtain the desired relation (1) from (7).

Theorem 2′. *If there exists a faithful, finitely generated, projective, injective left A-module L, then we have*

(1)′ $$\mathrm{Hom}_A([\mathrm{Ext}^n_A({}_A X, {}_A A)]_A, E(A_A)) = 0, \qquad n = 1, 2, \cdots$$

for every left A-module X.

Proof. Since L is finitely generated and projective, by [8, Lemma 7.1] there is an isomorphism

$$\mathrm{Hom}_A({}_A X, A)\otimes L \cong \mathrm{Hom}_A({}_A X, L),$$

which is natural in a left A-module X. Hence the above proof of Theorem 2 can be applied to the case where (4) is a projective resolution of a left A-module X. Hence we have (1)′.

Remark. Theorem 2′ is a generalization of [9, Theorem 4.1].

3. Proof of Theorem 4. Let Q be the double centralizer of $E({}_A A)$. Then ${}_A[E({}_Q Q)] = E({}_A A)$ by Lambek [6]. By assumption, ${}_Q Q$ is an injective cogenerator in the category ${}_Q\mathfrak{M}$ of all left Q-modules. Hence $E({}_Q Q) = {}_Q Q$ and ${}_A Q$ is an injective left A-module containing A

(cf. [6]). Furthermore, the double centralizer of ${}_AQ$ coincides with Q. Thus, applying our Theorem 7.1 in [10] we see that ${}_QQ_A \otimes_A Q_Q \cong {}_QQ_Q$. Since by Osofsky [11] (cf. also Kato [5]) $E(Q_Q)$ is an injective cogenerator in the category $\mathfrak{M}_Q$ and ${}_AQ \cong E({}_AA)$, by using [10, Theorem 7.1] again we can conclude that $[E(Q_Q)]_A$ is injective if and only if $E({}_AA)$ is flat.

Suppose that Q is also a maximal right quotient ring of A. Then $E(A_A) = [E(Q_Q)]_A$ and hence $[E(Q_Q)]_A$ is injective.

Conversely, suppose that $[E(Q_Q)]_A$ is injective. Then $[E(Q_Q)]_A \cong E(A_A) \oplus V_A$ for some right A-module V_A. By [10, Theorem 7.1] the double centralizer of $[E(Q_Q)]_A$ coincides with Q. Therefore, Q is contained in a maximal right quotient ring Q' of A. Since Q' is a rational extension of A, Q' is also a rational extension of Q. But the maximal right quotient ring of Q coincides with Q itself since $E(Q_Q)$ is a finitely cogenerating, injective cogenerator in $\mathfrak{M}_Q$ (cf. [10]). Hence $Q = Q'$, that is, Q is a maximal right quotient ring of A.

Thus, we have shown that Q is a maximal right quotient ring of A if and only if $[E(Q_Q)]_A$ is injective. As was shown before, the latter occurs if and only if $E({}_AA)$ is flat. Therefore, our Theorem 4 is completely proved.

Added in proof. At the autumn meeting of Math. Soc. Japan in 1970, it was announced by T. Kato that he obtained Theorem 2′ independently.

References

[1] H. Cartan and S. Eilenberg: Homological Algebra. Princeton University Press (1956).

[2] R. R. Colby and E. A. Rutter, Jr.: QF-3 rings with zero singular ideal. Pacific J. Math., **28**, 303–308 (1969).

[3] M. Harada: QF-3 and semi-primary *PP* rings. II. Osaka J. Math., **3**, 21–27 (1966).

[4] J. P. Jans: Duality in Noetherian rings. Proc. Amer. Math. Soc., **12**, 829–835 (1961).

[5] T. Kato: Torsionless modules. Tohoku Math. J., **20**, 234–243 (1968).

[6] J. Lambek: On Utumi's ring of quotients. Canad. J. Math., **15**, 363–370 (1963).

[7] K. Masaike: Quasi-Frobenius maximal quotient rings (to appear).

[8] K. Morita: Adjoint pairs of functors. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, **9**, No. 205, 40–71 (1965).

[9] ——: Duality in QF-3 rings. Math. Zeitschr., **108**, 237–252 (1969).

[10] ——: Localizations in categories of modules. I. Math. Zeitschr., **114**, 121–144 (1970).

[11] B. L. Osofsky: A generalizatoin of quasi-Frobenius rings. J. Algebra, **4**, 373–387 (1966).

[12] H. Tachikawa: On left QF-3 rings. Pacific J. Math., **32**, 255–268 (1970).

[13] R. M. Thrall: Some generalizations of quasi-Frobenius algebras. Trans. Amer. Math. Soc., **64**, 173–183 (1948).