212. *Notes on Regular Semigroups. II*

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First we give a new characterization of regular semigroups.[1)]

Theorem 1. *A semigroup S is regular if and only if the relation*

(1) $$L \cap R = RSL$$

holds for every left ideal L and every right ideal R of S.

Proof. Let S be a regular semigroup. Then the well known characterization due to L. Kovács and K. Iséki implies that

(2) $$L = SL$$

for any left ideal L of S, and similarly we have

(3) $$R = RS$$

for any right ideal R of S. (2) and (3) imply

(4) $$L \cap R = SL \cap RS = (RS)(SL) = RSL,$$

i.e., the condition (1) is necessary.

Conversely, let S be a semigroup with property (1) for any left ideal L and any right ideal R of S. To show that S is regular, let a be an arbitrary element of S. Then (1) implies

(5) $$a \in L(a) \cap R(a) = R(a)SL(a) \subseteq aSa,$$

that is, S is a regular semigroup.

Next we give a similar characterization of semigroups which are semilattices of groups.[2)]

Theorem 2. *A semigroup S is a semilattice of groups if and only if the relation*

(6) $$L \cap R = LSR$$

holds for every left ideal L and every right ideal R of S.

Proof. Let S be a semigroup which is a semilattice of groups. It is known that every one-sided ideal of S is two-sided and S is regular (see [1] or [4]). This implies that

(7) $$SI = I = IS$$

holds for any ideal I of S. Hence we get

(8) $$I_1 \cap I_2 = I_1 S \cap SI_2 = I_1 S I_2$$

for any couple of (two-sided) ideals of S, i.e. the condition (6) holds.

Conversely, let S be a semigroup with property (6) for any left ideal L and any right ideal R of S. Then (6) implies that $L = LS^2$ and

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1) For the notation and terminology we refer to [1].

2) For other characterizations of semigroups which are semilattices of groups, see [3]–[5].

$R=S^2R$ for any left ideal L and any right ideal R of S, respectively. Hence every one-sided ideal of S is two-sided, that is, S is a duo semigroup. On the other hand (6) implies

$$ISI=I \tag{9}$$

for any two-sided ideal I of S. But every two-sided ideal is a quasi-ideal of S, and conversely. Hence a result of Luh [7] guarantees that S is regular. By Theorem 3 of the first part of this note, S is a semilattice of groups.

Analogously can be proved the following result, utilizing that every bi-ideal is a two-sided ideal in a semigroup which is a semilattice of groups (see the author [6]).

Theorem 3. *For a semigroup S the following statements are equivalent and any one of them is a necessary and sufficient condition concerning S to be a semilattice of groups:*

1	B^3	18	BSQ
2	B^2Q	19	QSB
3	BQB	20	QSQ
4	QB^2	21	LSB
5	BQ^2	22	LSQ
6	QBQ	23	BSR
7	Q^2B	24	QSR
8	Q^3	25	LSR
9	LB^2	26	BIB
10	LBQ	27	BIQ
11	LQB	28	QIB
12	LQ^2	29	QIQ
13	B^2R	30	LIB
14	BQR	31	LIQ
15	QBR	32	BIR
16	Q^2R	33	QIR
17	BSB	34	LIR

Remark. B, I, L, Q, and R denote bi-, two-sided, left, quasi-, and right ideal of S, respectively. For example, the statement 10 means that S is a semigroup with property $L\cap B\cap Q=LBQ$ for any bi-ideal B, any left ideal L, and any quasi-ideal Q of S. The Condition 1 means that

$$\bigcap_{i=1}^{3} B_i=\prod_{i=1}^{3} B_i \tag{10}$$

holds for any triple B_1, B_2, B_3 of bi-ideals of S.

References

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