19. *The Implicational Fragment of R-mingle*

By Saburo TAMURA
Department of Mathematics, Yamaguchi University

(Comm. by Kinjirô KUNUGI, M. J. A., Jan. 12, 1971)

The relevant logic R was first defined in Belnap [1] though the implicational fragment of R which we refer to as RI in this note goes back to Church's weak implication [2]. Kripke [3] constructed "Sequenzen-kalkül" equivalent to RI. Anderson and Belnap [4] and the author [5] gave systems of the natural deduction equivalent to RI. By adding a mingle axiom $\alpha\supset(\alpha\supset\alpha)$ to R, we get a system R-mingle RM (defined by Meyer and Dunn [6]). Here the mingle axiom has the effect of Gentzen type "mingle" rule introduced by Ohnishi and Matsumoto [7].

In this note we shall give a system of the natural deduction equivalent to RMI, that is, the implicational fragment of RM. And then we shall show that the cut elimination theorem holds in Sequenzen-kalkül equivalent to RMI. Finally we shall give the decision procedure for RMI.

(A) The calculus RMI.

(Aa) **Axioms.**

Let α, β, γ be arbitrary formulae.

(Aa1) $((\alpha\supset\alpha)\supset\beta)\supset\beta$.

(Aa2) $(\alpha\supset\beta)\supset((\beta\supset\gamma)\supset(\alpha\supset\gamma))$.

(Aa3) $(\alpha\supset(\alpha\supset\beta))\supset(\alpha\supset\beta)$.

(Aa4) $\alpha\supset((\alpha\supset\alpha)\supset\alpha)$.

(Aa5) $\alpha\supset(\alpha\supset\alpha)$.

(Ab) **Provability.**

(Ab1)–(Ab5) Each of the axioms, (Aa1)–(Aa5), is provable in RMI.

(Ab6) If α and $\alpha\supset\beta$ are provable in RMI, then β is provable in RMI. This rule is called modus ponens (MP).

We shall abbreviate the statement "α is provable (in RMI)" to "$(RMI)\vdash\alpha$".

(Ac) **Derived rules and theorems.**

Let $A_n(\xi)$ denote the formula $\alpha_n\supset(\cdots\supset(\alpha_1\supset\xi)\cdots)$, where $A_0(\xi)$ means the formula ξ. Let $B_m(\xi)$ denote $\beta_m\supset(\cdots\supset(\beta_1\supset\xi)\cdots)$, where $B_0(\xi)$ means ξ.

(Ac1) $\vdash\alpha\supset\alpha$.

(Ac2) If $\vdash\alpha\supset\beta$ and $\vdash\beta\supset\gamma$, then $\vdash\alpha\supset\gamma$.

(Ac3) If $\vdash\alpha\supset\beta$ and $\vdash\gamma\supset((\alpha\supset\beta)\supset\delta)$, then $\vdash\gamma\supset\delta$.

(Ac4) If $\vdash \alpha \supset \beta$, then $\vdash A_n(\alpha) \supset A_n(\beta)$.
(Ac5) If $\vdash \alpha \supset \beta$ and $\vdash A_n(\alpha)$, then $\vdash A_n(\beta)$.
(Ac6) $\vdash \alpha \supset ((\alpha \supset \beta) \supset \beta)$.
(Ac7) $\vdash (\alpha \supset (\beta \supset \gamma)) \supset (\beta \supset (\alpha \supset \gamma))$.
(Ac8) $\vdash A_n(\alpha \supset \beta)$ if and only if $\vdash \alpha \supset A_n(\beta)$.
(Ac9) $\vdash A_n(B_m(\alpha))$ if and only if $\vdash B_m(A_n(\alpha))$.

(B) The calculus *NRMI*.

(Ba) **Inference rules.**

Let α, β be arbitrary formulae.

(Ba1) $\dfrac{\alpha \quad \alpha}{\alpha}$. This rule is called a mingle.

(Ba2) $\dfrac{\begin{array}{c}[\alpha]\\ \hline\hline \beta\end{array}}{\alpha \supset \beta}$. In this rule, which is called an $\supset$-*I*, assumption formulae α must actually occure above the formula β.

(Ba3) $\dfrac{\alpha \quad \alpha \supset \beta}{\beta}$. This rule is called an $\supset$-*E*.

(Bb) **Dependence and provability.**

(Bb1) In the rule (Ba1), the lower formula α depends on assumptions of the upper formulae α.

(Bb2) In the rule (Ba2), $\alpha \supset \beta$ depends on assumptions, except α, on which β depends.

(Bb3) In the rule (Ba3), β depends on assumptions of α and $\alpha \supset \beta$.

(Bb4) The assumption formula depends on itself.

(Bb5) The formula which depends on no assumption is called provable in *NRMI*.

We shall abbreviate the statement "α is provable (in *NRMI*)" to "$(NRMI) \vdash \alpha$".

(C) The calculus *LRMI*.

(Ca) **Inference rules.**

Let $\alpha, \beta, \gamma, \delta$ be arbitrary formulae, Γ, Σ be arbitrary (possibly empty) finite series of formulae separated by commas.

(Ca1) $\alpha \rightarrow \alpha$.

(Ca2) $\dfrac{\alpha, \alpha, \Gamma \rightarrow \delta}{\alpha, \Gamma \rightarrow \delta}$. This rule is called a contraction ($c\rightarrow$).

(Ca3) $\dfrac{\Sigma, \alpha, \beta, \Gamma \rightarrow \delta}{\Sigma, \beta, \alpha, \Gamma \rightarrow \delta}$. This rule is called an interchange ($i\rightarrow$).

(Ca4) $\dfrac{\Sigma \rightarrow \delta \quad \Gamma \rightarrow \delta}{\Sigma, \Gamma \rightarrow \delta}$. This rule is called a mingle (m).

(Ca5) $\dfrac{\Sigma \rightarrow \gamma \quad \gamma, \Gamma \rightarrow \delta}{\Sigma, \Gamma \rightarrow \delta}$. This rule is called a cut about $\gamma(\gamma)$.

(Ca6) $\dfrac{\Sigma \rightarrow \alpha \quad \beta, \Gamma \rightarrow \delta}{\alpha \supset \beta, \Sigma, \Gamma \rightarrow \delta}$. This rule is called an $\supset$-introduction in the antecedent ($\supset\rightarrow$).

(Ca7) $\dfrac{\alpha, \Gamma\to\beta}{\Gamma\to\alpha\supset\beta}$. This rule is called an $\supset$-introduction in the succedent ($\to\supset$).

(Cb) **Provability.**

(Cb1) Any sequent of the form (Ca1) is provable in *LRMI*.

(Cb2)–(Cb7) If every upper sequent in each of the rules, (Ca2)–(Ca7), is provable in *LRMI*, then the lower sequent in the rule is provable in *LRMI*. We shall abbreviate the statement "$\Gamma\to\alpha$ is provable (in *LRMI*)" to "(*LRMI*)$\vdash\Gamma\to\alpha$".

(D) The equivalence of RMI, NRMI and LRMI.

(Da) *If RMI*$\vdash\alpha$, *then NRMI*$\vdash\alpha$.

(Aa1): *NRMI*$\vdash((\alpha\supset\alpha)\supset\beta)\supset\beta$.

This is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\alpha}\quad\overset{1}{\alpha}}{\alpha}}{\alpha\supset\alpha}{\scriptstyle 1}\quad (\alpha\supset\overset{2}{\alpha})\supset\beta}{\beta}}{((\alpha\supset\alpha)\supset\beta)\supset\beta}{\scriptstyle 2}.$$

(Aa2)–(Aa4):

These are easily proved along the line of Gentzen [8] (see [5]).

(Aa5): *NRMI*$\vdash\alpha\supset(\alpha\supset\alpha)$.

This is transformed into:

$$\dfrac{\dfrac{\dfrac{\overset{1}{\alpha}\quad\overset{2}{\alpha}}{\alpha}}{\alpha\supset\alpha}{\scriptstyle 1}}{\alpha\supset(\alpha\supset\alpha)}{\scriptstyle 2}.$$

(MP):

This is easily proved by $\supset$-*E* in *NRMI*.

(Db) *If NRMI*$\vdash\alpha$, *then LRMI*$\vdash\to\alpha$.

(Ba1): If *LRMI*$\vdash\Sigma\to\alpha$ and *LRMI*$\vdash\Gamma\to\alpha$, then *LRMI*$\vdash\Sigma, \Gamma\to\alpha$.

This is easily proved by a mingle in *LRMI*.

(Ba2)–(Ba3):

These are easily proved along the line of Gentzen [8] (see [5]).

(Dc) *If LRMI*$\vdash\alpha_1, \cdots, \alpha_n\to\alpha$, *then RMI*$\vdash A_n(\alpha)$, *where* $A_n(\alpha)$ *is defined as* $\alpha_n\supset(\cdots\supset(\alpha_1\supset\alpha)\cdots)$.

(Ca1): *RMI*$\vdash\alpha\supset\alpha$.

This is evident by (Ac1).

(Ca2): If *RMI*$\vdash A_n(\alpha\supset(\alpha\supset\delta))$, then *RMI*$\vdash A_n(\alpha\supset\delta)$.

We can prove this by using (Aa3) and (Ac5).

(Ca3): If *RMI*$\vdash A_n(\beta\supset(\alpha\supset B_m(\delta)))$, then *RMI*$\vdash A_n(\alpha\supset(\beta\supset B_m(\delta)))$.

We can prove this by using (Ac7) and (Ac5).

(Ca4): If *RMI*$\vdash B_m(\delta)$ and *RMI*$\vdash A_n(\delta)$, then *RMI*$\vdash A_n(B_m(\delta))$.

This is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Aa5})}{\delta\supset(\delta\supset\delta)}\quad A_n(\delta)}{A_n(\delta\supset\delta)}{\scriptstyle(\mathrm{Ac5})}}{\delta\supset A_n(\delta)}{\scriptstyle(\mathrm{Ac8})}\quad B_m(\delta)}{B_m(A_n(\delta))}{\scriptstyle(\mathrm{Ac5})}}{A_n(B_m(\delta))}{\scriptstyle(\mathrm{Ac9})}$$

(Ca5): If $RMI\vdash B_m(\gamma)$ and $RMI\vdash A_n(\gamma\supset\delta)$, then $RMI\vdash A_n(B_m(\delta))$.

This is transformed into:

$$\dfrac{\dfrac{\dfrac{A_n(\gamma\supset\delta)}{\gamma\supset A_n(\delta)}{\scriptstyle(\mathrm{Ac8})}\quad B_m(\gamma)}{B_m(A_n(\delta))}{\scriptstyle(\mathrm{Ac5})}}{A_n(B_m(\delta))}{\scriptstyle(\mathrm{Ac9})}$$

(Ca6): If $RMI\vdash B_m(\alpha)$ and $RMI\vdash A_n(\beta\supset\delta)$, then $RMI\vdash A_n(B_m((\alpha\supset\beta)\supset\delta))$.

This is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(\mathrm{Ac6})}{\alpha\supset((\alpha\supset\beta)\supset\beta)}\quad B_m(\alpha)}{B_m((\alpha\supset\beta)\supset\beta)}{\scriptstyle(\mathrm{Ac5})}}{(\alpha\supset\beta)\supset B_m(\beta)}{\scriptstyle(\mathrm{Ac8})}\qquad \dfrac{\dfrac{A_n(\beta\supset\delta)}{\beta\supset A_n(\delta)}{\scriptstyle(\mathrm{Ac8})}}{B_m(\beta)\supset B_m(A_n(\delta))}{\scriptstyle(\mathrm{Ac4})}}{(\alpha\supset\beta)\supset B_m(A_n(\delta))}{\scriptstyle(\mathrm{Ac2})}}{B_m(A_n((\alpha\supset\beta)\supset\delta))}{\scriptstyle(\mathrm{Ac8})}}{A_n(B_m((\alpha\supset\beta)\supset\delta))}{\scriptstyle(\mathrm{Ac9})}$$

(Ca7): If $RMI\vdash A_n(\alpha\supset\beta)$, then $RMI\vdash A_n(\alpha\supset\beta)$.

(E) The cut-elimination theorem in LRMI.

(Ea) Theorem.

If $LRMI\vdash\Gamma\to\delta$, then $\Gamma\to\delta$ is provable without cuts in LRMI.

Proof. The proof is treated along the line of Gentzen [8]. We shall here consider a rule called a fusion (cf. [7]), which is expressed by the following form:

$$\frac{\Gamma\to\gamma\quad \Sigma^n\to\delta}{\Gamma,\Sigma^{n'}\to\delta}(\gamma)$$

where $n>n'\geq 0$ and $\Sigma^n(\Sigma^{n'})$ show finite series of formulae that include $n(n')$ formulae of the form γ.

We can easily prove that every fusion may be transformed into a cut by using several interchanges and contractions. Conversely every cut may be regarded as a special fusion. Then we have only to prove the following:

Lemma. *Any proof-figure with a fusion as its lowest rule and no other fusion over it can be transformed into a proof-figure with the same endsequent in which no fusion occures.*

Proof. The definitions of the degree and the rank of a fusion being the same as in Gentzen [8], the proof can be carried out by the double induction on the degree and the rank (see [5]).

(F) Decision procedure for RMI.

We can prove the rule called anti-contraction

$$\frac{\alpha, \Gamma \to \delta}{\alpha, \alpha, \Gamma \to \delta}$$

in *LRMI* as follows:

$$\cfrac{\cfrac{\alpha \to \alpha \quad \alpha \to \alpha}{\alpha, \alpha \to \alpha}{\scriptstyle (m)} \qquad \alpha, \Gamma \to \delta}{\alpha, \alpha, \Gamma \to \delta}{\scriptstyle (\alpha)}$$

Thus we can prove that *LRMI* has a decision procedure in the same way as Gentzen did in [8]. Therefore *RMI* has a decision procedure.

References

[1] N. D. Belnap: Intensional models for first degree formulas. Journal of Symbolic Logic, **32**, 1–22 (1967).
[2] A. Church: The Weak Theory of Implication. Kontrolliertes Denken. Munichi (1951).
[3] S. A. Kripke: The problem of entailment. Journal of Symbolic Logic, **24**, 324 (1959).
[4] A. R. Anderson and N. D. Belnap: The pure calculus of entailment. Journal of Symbolic Logic, **27**, 19–52 (1962).
[5] S. Tamura: On the weak implication calculus. Journal of the Faculty of Liberal Arts, Yamaguchi University, **4**, 75–83 (1970).
[6] R. K. Meyer and J. M. Dunn: E, R and γ. Journal of Symbolic Logic, **34**, 460–474 (1969).
[7] M. Ohnishi and K. Matsumoto: A system for strict implication. Annals of the Japanese Association for Philosophy of Science, **2**, 183–188 (1964).
[8] G. Gentzen: Untersuchungen über das logische Schließen. Mathematische Zeitschrift, **39**, 176–210, 405–431 (1934).