14. A Simple Geometric Construction of Weakly Mixing Flows which are not Strongly Mixing

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1. The existence of measure preserving transformations which are weakly but not strongly mixing has been discussed by Halmos [4], Kakutani-von Neumann [5] and Chacon [1], [2], [3]. Maruyama [6] has shown the existence of Gaussian flows of this type by some results in Gaussian processes. In this short paper we shall give a general method for constructing flows of the type, of which idea is obtained from Chacon [2], [3].

2. Let $(\Omega, \mathcal{L}, \mu)$ be a Lebesgue space, where $\Omega=[0,1)\times[0,1)$, $\mathcal{L}$ is the product Lebesgue class and μ is the usual product Lebesgue measure defined on $\mathcal{L}$.

Definition 1. A flow $\{T_t\}$ on $(\Omega, \mathcal{L}, \mu)$ is said to be ergodic if there exists a positive number t such that $\mu(T_tA\cap B)>0$ holds for every pair A, B from $\mathcal{L}$ with positive measure.

Definition 2. If there exist a complex number λ with the absolute value one and a function f in $L^2(\Omega)$ such that

$$f(T_t(x, y))=\lambda^t f(x, y) \qquad \text{for a.a. } (x, y)\in\Omega \text{ and all } t,$$

we call λ and f an eigenvalue and an eigenfunction corresponding to λ, respectively.

Definition 3. A flow $\{T_t\}$ is weakly mixing if the flow cannot have simple eigenvalues other than one.

Definition 4. A flow $\{T_t\}$ is strongly mixing if

$$\lim_{t\to\infty}\mu(T_tA\cap B)=\mu(A)\mu(B)$$

holds for every pair A, B from $\mathcal{L}$ with positive measure.

Definition 5. For a set A of $\mathcal{L}$ with positive measure, a local flow φ_t on A is defined as follows:

$$\varphi_t(x, y)=\begin{cases}(x, y+t) & \text{if } (x, y+t)\in A,\\ \text{undefined elsewhere,}\end{cases}$$

for each $(x, y)\in A$.

Our main result may be stated as follows:

Theorem. *There exists a weakly mixing flow $\{T_t\}$ on $(\Omega, \mathcal{L}, u)$ which is not strongly mixing.*

Proof. After the flow is constructed, we will prove that it is weakly but not strongly mixing using a direct argument. The first step of the construction is the following. We divide the two dimensional torus Ω into three pairwise disjoint and consecutive rectangles represented by R_{11}, R_{12} and Q_1, where $R_{11}=[0,1/3)\times[0,1)$, $R_{12}=[1/3,2/3)\times[0,1)$ and $Q_1=[2/3,1)\times[0,1)$. Put R_{12} on R_{11} identifying the points $(0,1)$ and $(1/3,1)$ with the points $(1/3,0)$ and $(2/3,0)$, respectively. We define the local flow $\varphi_t^{(1)}$ on the set R_1, where $R_1=[0,1/3)\times[0,2)$. For later convenience' sake, we divide R_1 into some squares, $A_{11}=[0,1/3)\times[0,1/3), \cdots, A_{16}=[0,1/3)\times[5/3,2)$ and denote by $\mathfrak{A}_1$ the family $\{A_{11}, \cdots, A_{16}, Q_1\}$. Take

$$p(1)=2 \quad \text{and} \quad \sigma(1)=6.$$

Next, we suppose that the n-1th step has been already constructed. In an analogous method to the above, we divide R_{n-1} and Q_{n-1} into two pairwise disjoint and consecutive rectangles $R_{n1}, R_{n2}: Q_{n1}, Q_{n2}$, respectively. Put R_{n2} on R_{n1} and Q_{n1} on R_{n2}, and denote by R_n the set, that is, $R_n=[0,1/3(1/2)^{n-1})\times[0,p(n))$, and take $Q_n=Q_{n2}$. We define the local flow $\varphi_t^{(n)}$ on the set R_n. Furthermore, divide R_n into some pairwise disjoint and consecutive squares, $A_{n1},\cdots,A_{n\sigma(n)}$ and put $\mathfrak{A}_n=\{A_{n1},\cdots, A_{n\sigma(n)}, Q_n\}$. Here, the squares $A_{nk}, 1\leqq k\leqq\sigma(n)$, have the same area. A simple calculation shows that for $n\geqq 2$,

$$p(n)=2\cdot p(n-1)+1 \quad \text{and} \quad \sigma(n)=4\cdot\sigma(n-1)+2^{n-2}\cdot 6.$$

It is clear from the geometric interpretation that $\varphi_t^{(n+1)}=\varphi_t^{(n)}$ on the domain of the definition of $\varphi_t^{(n)}$, and that $\lim_n \varphi_t^{(n)}$ exists almost everywhere. Indeed, this limit, $\lim_n \varphi_t^{(n)}$, is the common extension of $\varphi_t^{(n)}$ for all n. Let

$$T_t=\lim_{n\to\infty}\varphi_t^{(n)}.$$

Obviously, the flow $\{T_t\}$ is ergodic. Noticing that $\mu(\varphi_n^{(n)}R_{11}\cap R_{11})=1/3(1/2)^{n-1}$ holds for all $n\geqq 2$, and putting $T=T_1$, it is easily verified that

$$\lim_{n\to\infty}\mu(T^nR_{11}\cap R_{11})\neq\{\mu(R_{11})\}^2=\frac{1}{9}$$

This shows that the flow $\{T_t\}$ is not strongly mixing.

In what follows we prepare the following lemmas which are essential for our purpose.

Lemma 1. *For any positive number ε and for any Borel set B with positive measure, there exist an integer n and a subset α of* $(1,2,\cdots,\sigma(n))$ *such that*

$$\mu(B\ominus\bigcup_{k\in\alpha}A_{nk})<\varepsilon \quad \text{and} \quad \mu(B\cap A_{nk})\geqq(1-\varepsilon)\mu(A_{nk})$$

for $k\in\alpha$, where $\mathfrak{R}_n=\{A_{n1},\cdots,A_{n\sigma(n)},Q_n\}$.

Proof. This is easily obtained from the strong density theorem of S. Saks (see Saks [7]).

Lemma 2. *For any positive number ε and for any Borel function f, there exist an integer n and a subset α of* $(1,2,\cdots,\sigma(n))$ *such that*

*f is simple within ε on $\mathfrak{A}_\alpha=\{A_{nk}, k\in\alpha\}$, where $\mu(\bigcup_{k\in\alpha}A_{nk})\geqq(1-\varepsilon)$, that is, f is constant within ε on each A_{nk} for $k\in\alpha$.**)

Proof. It follows at once from Lemma 1.

It remains to prove that $\{T_t\}$ is weakly mixing. To this end, we suppose that the flow $\{T_t\}$ has an eigenfunction f such that

$$f(T_t(x,y))=\lambda^t f(x,y) \qquad \text{for a.a. } (x,y)\in\Omega \text{ and all } t$$

Moreover, we may assume without loss of generality that $|f|\geqq K$ a.e. for some positive number K. It follows from Lemma 2 that for any positive number ε, there exist an integer n and a subset α of $(1,2,\cdots,\sigma(n))$ such that f is simple within ε on $\mathfrak{R}_\alpha=\{A_{nk}, k\in\alpha\}$. Now consider such a set A_{nk} for a $k(1\leqq k\leqq p(n)-1)$ fixed in α. Then, by virtue of the manner of the construction of $\{T_t\}$, one can easily verify that

$$\mu(T^{p(n)}A_{nk}\cap A_{nk})=\frac{1}{2}\mu(A_{nk}) \quad \text{and} \quad \mu(T^{p(n)+1}A_{nk}\cap A_{nk})=\frac{1}{4}\mu(A_{nk})$$

$(T=T_1)$. Let $c(\varepsilon)$ be the constant approximating f on A_{nk} with an error ε:

$$|f(x,y)-c(\varepsilon)|\leqq\varepsilon \qquad \text{on } E_{nk}$$

where E_{nk} is a subset of A_{nk} such that $\mu(E_{nk})\geqq(1-\varepsilon)\mu(A_{nk})$. Then, by Lemma 2, there exists a positive constant $\delta(f)$ which satisfies

$$|c(\varepsilon)|\geqq\delta(f).$$

If we let $(x,y)\in T^{p(n)}A_{nk}\cap A_{nk}$, then we have

$$f(T^{p(n)}(x,y))=\lambda^{p(n)}f(x,y).$$

If we let $(x,y)\in T^{p(n)+1}A_{nk}\cap A_{nk}$, then we have

$$f(T^{p(n)+1}(x,y))=\lambda^{p(n)+1}f(x,y).$$

From the above relations, one obtains that

$$|\lambda^{p(n)}c(\varepsilon)-c(\varepsilon)|\leqq 2\varepsilon \quad \text{and} \quad |\lambda^{p(n)+1}c(\varepsilon)-c(\varepsilon)|\leqq 2\varepsilon$$

from which it follows that $\lambda=1$. This completes the proof.

References

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*) Following Chacon [2], we say that for a positive number ε and for a measurable set A, a function f is constant within ε on A if there exist a constant c and a measurable subset E of A such that $|f(z)-c|\leqq\varepsilon$ on E and such that $\mu(E)\geqq(1-\varepsilon)\mu(A)$. Let $\mathfrak{A}=\{A_1,\cdots,A_n\}$ be a class of pairwise disjoint measurable sets. We say that a function f is simple within ε on $\mathfrak{A}$ if it is constant within ε on each A_k, $1\leqq k\leqq n$ (the n constants are not necessarily equal).

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