13. *Some Radii Associated with Polyharmonic Equations*

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Introduction. G. Pólya and G. Szegö [2] defined the inner radius of a bounded domain by a conformal correspondence from the domain to a disk and showed that it can be also given by the Green's function of the domain relative to the Laplace's equation $\Delta u=0$. In addition, they defined the biharmonic inner radius of a domain by the Green's function of the domain concerning the biharmonic equation $\Delta^2 u=0$. Using the results, they calculated the ordinary inner and biharmonic inner radii of a nearly circular domain. The aim of this paper is to extend the above results. In the first place, we obtain the Green's function of a disk relative to the n-harmonic equation $\Delta^n u=0$ and define the n-harmonic inner radius of a domain. On the base of the results, we compute the n-harmonic inner radius of a nearly circular domain and it is remarkable that it is monotonously decreasing with respect to integer n.

1. Inner radii associated with polyharmonic equations.

We use the following notations hereafter. Let D be a bounded domain, C the boundary of D, a an inner point of D, z the variable point in D and r the distance from a to z.

Definition 1. The function satisfying following two conditions is called the Green's function of D with the pole a relative to the n-harmonic equation $\Delta^n u=0$.

(1) The function has in a neighborhood of a the form

$$r^{2(n-1)}\log r+h_n(z),$$

where the function $h_n(z)$ satisfies the equation $\Delta^n u=0$ in D.

(2) On the boundary C, the function and all its normal derivatives of order $\leqq n-1$ vanish.

Theorem 1. *If D is the disk $|z|<R$ in the complex z-plane, the Green's function $G_n(a,z)$ of D with the pole a relative to the equation $\Delta^n u=0$ is as follows,*

$$G_n(a,z)=|z-a|^{2(n-1)}\log\left|\frac{R(z-a)}{R^2-\bar{a}z}\right|$$
$$-\frac{1}{2}\sum_{k=1}^{n-1}\frac{|z-a|^{2(n-k-1)}}{kR^{2k}}\{|R(z-a)|^2-|R^2-\bar{a}z|^2\}^k.$$

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Proof. We rewrite the function $G_n(a, z)$ as

$$G_n(a, z)=|z-a|^{2(n-1)}\log|z-a|-|z-a|^{2(n-1)}\log\frac{|z-a|}{R}$$
$$-\frac{1}{2}\sum_{k=1}^{n-1}\frac{|z-a|^{2(n-k-1)}}{kR^{2k}}\{|R(z-a)|^2-|R^2-\bar{a}z|^2\}^k.$$

Then the part of the summation of the right hand side is obviously n-harmonic. By the fact that if the function u is harmonic, the function $r^{2(n-1)}u$ is n-harmonic, the second term of the right hand side of the former equality is also n-harmonic. So our function satisfies the condition (1) of the Green's function. Now, we put

$$x=\left|\frac{R^2-\bar{a}z}{R(z-a)}\right|^2.$$

Then x is equal to 1 on the boundary $|z|=R$, and we can rewrite the function $G_n(a, z)$ as

$$G_n(a, z)=-\frac{1}{2}r^{2(n-1)}\left\{\log x+\sum_{k=1}^{n-1}\frac{1}{k}(1-x)^k\right\}.$$

And if $f_n(x)$ denotes the following function

$$\log x+\sum_{k=1}^{n-1}\frac{1}{k}(1-x)^k,$$

$f_n(1)$ and $f_n^{(\alpha)}(1)$, for such an integer α as $1\leqq\alpha\leqq n-1$, vanish. Consequently, we can verify the function $G_n(a, z)$ satisfies the condition (2) of the Green's function. That establishes the theorem.

Given a domain D and an inner point a of D, G. Pólya and G. Szegö [2] defined the inner radius r_a of D with respect to the point a as follows; The interior of D being mapped conformally onto the interior of a circle so that the point a corresponds to the center of the circle and linear magnification at the point a is equal to 1, the radius of the circle so obtained is r_a. When the Green's function $G(a, z)$ of D with the pole a relative to the equation $\Delta u=0$ is

$$G(a, z)=\log r-h(z),$$

they showed that the inner radius r_a is determined by

$$\log r_a=h(a).$$

They also defined the biharmonic inner radius associated with the biharmonic equation $\Delta^2 u=0$ as follows; Denoted the Green's function of D with the pole a relative to the biharmonic equation $\Delta^2 u=0$ by

$$r^2\log r+h_2(z)$$

and putting

$$\frac{s_a^2}{2}=h_2(a),$$

the positive quantity s_a is called the biharmonic inner radius of D with respect to the point a.

Now we define the n-harmonic inner radius of a domain D associated with the n-harmonic equation $\Delta^n u=0$.

Definition 2. If the Green's function of a domain D with the pole a relative to the equation $\Delta^n u=0$ is

$$r^{2(n-1)}\log r+h_n(z),$$

and we put

$$\log r_{a,1}=-h_1(a),$$
$$\frac{r_{a,n}^{2(n-1)}}{2(n-1)}=|h_n(a)| \qquad (n\geqq 2),$$

we call the positive quantity $r_{a,n}$ the n-harmonic inner radius of the domain D with respect to the point a.

Remark. When the domain D is the disk $|z|<R$ in the complex z-plane, it is well known that the Green's function of D with the pole a relative to the equation $\Delta u=0$ is

$$\log\left|\frac{R(z-a)}{R^2-\bar{a}z}\right|,$$

and the Green's function relative to the equation $\Delta^2 u=0$ has been given by G. Pólya and G. Szegö as follows,

$$|z-a|^2\log\left|\frac{R(z-a)}{R^2-\bar{a}z}\right|-\frac{1}{2R^2}\{|R(z-a)|^2-|R^2-\bar{a}z|^2\}.$$

Using the preceding two Green's functions and the Green's function given in Theorem 1, we can obtain the ordinary inner radius, the biharmonic inner radius and the n-harmonic inner radius for an arbitrary integer n $(n\geqq 3)$ of the disk $|z|<R$ with respect to the point a, which are the same value

$$\frac{R^2-|a|^2}{R}.$$

2. Inner radii of a nearly circular domain.

In this section, we treat the radii of a nearly circular domain defined in the former section.

Definition 3. Let

$$(1) \qquad r=1+\rho(\varphi)$$

be the equation of the boundary of a domain in polar coordinate r and φ, where the periodic function $\rho(\varphi)$ represents the infinitesimal variation of the unit circle. We call the domain bounded by (1) the nearly circular domain.

We consider the Fourier series

$$(2) \qquad \rho(\varphi)=a_0+2\sum_{k=1}^{\infty}(a_k\cos k\varphi+b_k\sin k\varphi),$$

where each coefficient a_k or b_k is the infinitesimal of the first order. Terms of higher infinitesimal than the second order are neglected in all the discussions of this section.

Lemma. *Neglecting terms of higher than the first order, the centroid* $c=|c|e^{i\gamma}$ *of the nearly circular domain* $r<1+\rho(\varphi)$ *is*

$$(3) \qquad c=2(a_1+ib_1).$$

This lemma was given by G. Pólya and G. Szegö [2], and they obtained the ordinary inner radius r_c and the biharmonic inner radius s_c of the nearly circular domain with respect to the centroid c as follows,

$$(4)\qquad \begin{aligned} r_c&=1+a_0+a_1^2+b_1^2-\sum_{k=2}^{\infty}(2k+1)(a_k^2+b_k^2),\\ s_c&=1+a_0+a_1^2+b_1^2-\sum_{k=2}^{\infty}(4k-1)(a_k^2+b_k^2).\end{aligned}$$

As an extension of (4), we prove the following theorem.

Theorem 2. *For an arbitrary positive integer n, the n-harmonic inner radius $r_{c,n}$ of the nearly circular domain $r<1+\rho(\varphi)$ with respect to the centroid c is*

$$(5)\qquad r_{c,n}=1+a_0+a_1^2+b_1^2-\sum_{k=2}^{\infty}\{2nk-(2n-3)\}(a_k^2+b_k^2).$$

Consequently, $r_{c,n}$ decreases monotonously with respect to n.

Proof. We seek the Green's function $G_n(c,z)$ of the nearly circular domain with the pole c relative to the equation $\Delta^n u=0$ in the form

$$G_n(c,z)=r'^{2(n-1)}\log r'-\frac{1}{2}\sum_{k=1}^{n-1}\frac{r'^{2(n-k-1)}}{k}(r'^2-1)^k-p(r,\varphi)-q(r,\varphi),$$

$$p(r,\varphi)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}r^{k+2j}(A_{kj}\cos k\varphi+B_{kj}\sin k\varphi),$$

$$q(r,\varphi)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}r^{k+2j}(A'_{kj}\cos k\varphi+B'_{kj}\sin k\varphi),$$

where r' is the distance from the centroid $c=|c|e^{i\gamma}$ to the point $z=re^{i\varphi}$, the coefficients of $p(r,\varphi)$ are of the first order and those of $q(r,\varphi)$ of the second order. The n-harmonic inner radius $r_{c,n}$ is determined by

$$\frac{r_{c,n}^{2(n-1)}}{2(n-1)}=\left|(-1)^n\frac{1}{2(n-1)}-p(|c|,\gamma)-q(|c|,\gamma)\right|$$
$$=\frac{1}{2(n-1)}-(-1)^n\{A_{00}+(A_{10}\cos\gamma+B_{10}\sin\gamma)|c|+A'_{00}\},$$

and so we have

$$(6)\qquad r_{c,n}=1-(-1)^n\{A_{00}+(A_{10}\cos\gamma+B_{10}\sin\gamma)|c|+A'_{00}\}-\frac{2n-3}{2}A_{00}^2.$$

Setting

$$\lambda=r'^2\quad\text{and}\quad F_n(\lambda)=\frac{1}{2}\lambda^{n-1}\left\{\log\lambda-\sum_{k=1}^{n-1}\frac{1}{k}\left(1-\frac{1}{\lambda}\right)^k\right\},$$

we can rewrite as

$$G_n(c,z)=F_n(\lambda)-p(r,\varphi)-q(r,\varphi).$$

Owing to the equality

$$\lambda^{n-1}\left\{\frac{1}{\lambda}-\sum_{k=1}^{n-1}\frac{1}{\lambda^2}\left(1-\frac{1}{\lambda}\right)^{k-1}\right\}=\frac{1}{\lambda}(\lambda-1)^{n-1},$$

We obtain the following equality

$$(7)\qquad \frac{d}{d\lambda}F_n(\lambda)=(n-1)F_{n-1}(\lambda).$$

On the boundary $r=1+\rho(\varphi)$ we have

$$
(8)\quad
\begin{aligned}
F_1(\lambda)&=\log r'=\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}\\
&\qquad -\frac{1}{2}\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}^2+|c|^2\sin^2(\varphi-\gamma),\\
F_2(\lambda)&=r'\log r'-\frac{1}{2}(r'^2-1)=\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}^2,\\
F_\alpha(\lambda)&=0,\ 3\leqq\alpha;
\end{aligned}
$$

that is, when α is larger than 2, $F_\alpha(\lambda)$ is negligible. Let ν be the normal of the boundary of the nearly circular domain. Then the condition $\frac{\partial^m}{\partial\nu^m}F_n=0$ on the boundary can be replaced by $\frac{\partial^m}{\partial r^m}F_n=0$. On account of (7) and (8), we have

$$
\frac{\partial^{n-2}}{\partial r^{n-2}}F_n=(n-1)!\,F_2(\lambda)\left(\frac{\partial\lambda}{\partial r}\right)^{n-2},
$$

$$
\frac{\partial^{n-1}}{\partial r^{n-1}}F_n=(n-1)!\left\{F_1(\lambda)\left(\frac{\partial\lambda}{\partial r}\right)^{n-1}+\frac{1}{2}(n-1)(n-2)F_2(\lambda)\left(\frac{\partial\lambda}{\partial r}\right)^{n-3}\frac{\partial^2\lambda}{\partial r^2}\right\}.
$$

So the boundary conditions are

$$
(9)\quad
\begin{aligned}
&\frac{\partial^\alpha}{\partial r^\alpha}p(1,\varphi)+\rho(\varphi)\frac{\partial^{\alpha+1}}{\partial r^{\alpha+1}}p(1,\varphi)+\frac{\partial^\alpha}{\partial r^{\alpha+1}}q(1,\varphi)=0\quad 0\leqq\alpha\leqq n-3,\\
&\frac{\partial^{n-2}}{\partial r^{n-2}}p(1,\varphi)+\rho(\varphi)\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\varphi)+\frac{\partial^{n-2}}{\partial r^{n-2}}q(1,\varphi)\\
&\quad=2^{n-2}(n-1)!\,\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}^2,\\
&\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\varphi)+\rho(\varphi)\frac{\partial^{n}}{\partial r^{n}}p(1,\varphi)+\frac{\partial^{n-1}}{\partial r^{n-1}}q(1,\varphi)\\
&\quad=2^{n-1}(n-1)!\,\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}\\
&\qquad+2^{n-3}(n-1)!\,\{2(2n-3)+(n-1)(n-2)\}\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}^2\\
&\qquad+2^{n-2}(n-1)!\,|c|^2\sin^2(\varphi-\gamma).
\end{aligned}
$$

The first order terms yield

$$
(10)\quad
\begin{aligned}
&\frac{\partial^\alpha}{\partial r^\alpha}p(1,\varphi)=0\qquad 0\leqq\alpha\leqq n-2,\\
&\frac{\partial^{n-1}}{\partial r^{n-1}}p(1,\varphi)=2^{n-1}(n-1)!\,\{\rho(\varphi)-|c|\cos(\varphi-\gamma)\}.
\end{aligned}
$$

Noting that, by the first condition of (10), $p(r,\varphi)$ has the factor $(r^2-1)^{n-1}$ and in view of lemma we have

$$
|c|\cos(\varphi-\gamma)=2(a_1\cos\varphi+b_1\,\mathrm{bin}\,\varphi),
$$

we obtain the equality

$$
(11)\qquad p(r,\varphi)=(r^2-1)^{n-1}\left\{a_0+2\sum_{k=2}^{\infty}r^k(a_k\cos k\varphi+b_k\sin k\varphi)\right.;
$$

in particular

$$
(12)\qquad A_{00}=(-1)^{n-1}a_0\qquad\text{and}\qquad A_{10}=B_{10}=0.
$$

We consider the second order terms. The mean value of the function $q(r,\varphi)$ with respect to φ is equal to $\sum_{j=0}^{n-1}r^{2j}A'_{0j}$. By the first equation of (9) and that of (10) we have

$$\frac{\partial^\alpha}{\partial r^\alpha} q(1, \varphi)=0 \qquad 0 \leqq \alpha \leqq n-3,$$

so that it must be the form

$$\sum_{j=0}^{n-1} r^{2j} A'_{0j}=(r^2-1)^{n-2}(Ar^2+B), \tag{13}$$

where A and B are constants. Comparing the constant coefficients of $q(r, \varphi)$ and (13), we obtain

$$A'_{00}=(-1)^{n-2}B. \tag{14}$$

Taking now the mean values of second order terms we find

$$A+B=-(n-1)\left\{a_0^2+2\sum_{k=2}^{\infty}(a_k^2+b_k^2)\right\},$$

$$\begin{aligned}(n+2)A+(n-2)B\\ &=\{2(2n-3)-(n-1)(n+2)\}\left\{a_0^2+2\sum_{k=2}^{\infty}(a_k^2+b_k^2)\right\}\\ &\quad -8n\sum_{k=2}^{\infty}k(a_k^2+b_k^2)+4(a_1^2+b_1^2),\end{aligned}$$

and so we have

$$\begin{aligned}B=&-(a_1^2+b_1^2)+2n\sum_{k=2}^{\infty}k(a_k^2+b_k^2)\\ &-\frac{1}{2}(2n-3)\left\{a_0^2+2\sum_{k=2}^{\infty}(a_k^2+b_k^2)\right\}.\end{aligned} \tag{15}$$

By virtue of (6), (12), (14) and (15) we find

$$r_{c,n}=1+a_0+a_1^2+b_1^2-\sum_{k=2}^{\infty}\{2nk-(2n-3)\}(a_k^2+b_k^2).$$

This is the equality (5) of the theorem.

References

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