12. *On Characters of Singly Generated C^*-algebras*

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0. Let A be an operator in $B(H)$ the set of bounded operators on a Hilbert space H. Let $C^*(A)$ be the C^*-subalgebra of $B(H)$ generated by A and the identity. By a character on a C^*-algebra we mean a multiplicative linear functional.

The purpose of the present note is to investigate the existence of characters on $C^*(A)$ for some special operators.

Our result is a generalization of the following theorem of W. Arveson [2]: if $\lambda \in \sigma(A)$, $|\lambda|=\|A\|$ then there is a character ϕ on $C^*(A)$ such that $\phi(A)=\lambda$.

Our result is the following:

Theorem. *If A is a spectraloid operator and $\lambda \in \sigma(A)$ such that $|\lambda|=r(A)=\omega(A)$, then there is a character ϕ on $C^*(A)$ such that $\phi(A)=\lambda$.*

1. Let A be a bounded operator on H. A is called spectraloid if the spectral radius is equal to numerical norm i.e.,

$$\sup_{\lambda \in \sigma(T)} |\lambda| = \sup_{\mu \in W(A)} |\mu|$$

where

$$W(A)=\{\langle Ax, x\rangle, \|x\|=1\}.$$

First we give a more simple proof of Arveson result and next we generalize the construction to our case. Suppose that $\lambda \in a(A)=$ the approximate point spectrum of A. Then there exists a sequence $\{x_n\}$, $\|x_n\|=1$ such that

$$Ax_n-\lambda x_n \to 0$$

and since A is hyponormal

$$A^*x_n-\bar{\lambda}x_n \to 0.$$

Thus, for every polynomial $p(t,\bar{t})$ on $\sigma(A)$ we have

$$p(\lambda,\bar{\lambda}) \in \sigma(p(A,A^*))$$

Since $\{p(t,\bar{t})\}$ is dense in $C(\sigma(A))$ we obtain that

$$\phi(p(A,A^*))=p(\lambda,\bar{\lambda})$$

is a continuous linear functional on $C^*(A)$ which is clear multiplicative and thus the W. Arveson theorem is proved.

First we suppose that A is a spectraloid operator and the numerical range $W(A)=\{\langle Ax, x\rangle, \|x\|=1\}$ is a closed set, and also $a(A)=a_p(A)$

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where $a(A)$ is the approximate point spectrum and $a_p(A)$ the point spectrum. If $\xi \in \sigma(A)$ and

$$E_\xi(A) = \{x, Ax = \xi x\}$$

then ξ is called a normal eigenvalue if $E_\xi(A) = E_{\bar{\xi}}(A^*)$.

Since A is a spectraloid and if $\lambda \in \sigma(A)$ with $|\lambda| = r_A = \omega(A)$ it is clear that $\lambda \in \partial\sigma(A) \subset a(A) = a_p(A)$ which gives that $\lambda \in a_p(A) \cap \partial W(A)$. By a theorem of Hildebrandt (Theorem 2, [4]) λ is a normal eigenvalue. As in the case of hyponormal operators for all polynomials $p(t, \bar{t})$ on $\sigma(A)$ we have

$$p(\lambda, \bar{\lambda}) \in \sigma(p(A, A^*))$$

which shows that

$$\phi(p(A, A^*)) = p(\lambda, \bar{\lambda})$$

is a character of $C^*(A)$ and the theorem is proved for this case. The general case reduces to this by a construction of S. Berberian [1] of an extension K of H via a generalized limit. We omit the details.

Remark. It is a problem if the theorem holds for p-spectraloid operators considered by Furuta and Takeda [3].

References

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