10. On Semilattices of Groups. II

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This note is a continuation of, and is written in the same terminology as, the author's earlier paper [1]. There was proved the following result.

Theorem A. *A semigroup S is a semilattice of groups if and only if the intersection of any two bi-ideals of S equals to their product.*

This criterion has the following consequence.

Corollary. *Let S be a semilattice of groups. Then*

$$\bigcap_{i=1}^{k} B_i=\prod_{i=1}^{k} B_i \tag{1}$$

holds for any k bi-ideals $B_1, \cdots, B_k$ of S (k is an arbitrary fixed positive integer greater than one).

Here we show that property (1) is a necessary and sufficient condition for a semigroup S to be a semilattice of groups. First we prove this statement in case of $k=4$. The other cases can similarly be proved.

Theorem 1. *A semigroup S is a semilattice of groups if and only if the relation*

$$B_1 \cap B_2 \cap B_3 \cap B_4=B_1B_2B_3B_4 \tag{2}$$

holds for any four bi-ideals B_1, B_2, B_3, B_4 of S.

Proof. The necessity of the condition (2) is implied by the above Corollary of Theorem A.

Sufficiency. Let S be a semigroup with property (2) for every quadruplet of bi-ideals in S. Then (2) implies

$$B=SBS^2 \tag{3}$$

for every bi-ideal B of S, that is, every bi-ideal B of S is a two-sided ideal of S. To show that S is regular, let I be an arbitrary ideal of S. Then (2) implies

$$I=IS^2I. \tag{4}$$

Hence we have $I\subset ISI$ and $ISI\subset I$, because I is a two-sided ideal of S. Consequently

$$I=ISI \tag{5}$$

for any ideal I of S. This implies that S is regular (cf. Luh [5]). Therefore S is a regular duo semigroup, i.e. S is a semilattice of groups (by Theorem 3 in author's paper [2]).

Theorem 2. *A semigroup S is a semilattice of groups if and only if the condition* (1) *holds for any k bi-ideals $B_1, \cdots, B_k$ of S (k is a fixed* *positive integer greater than one).*

The proof of this assertion in case of any k is quite similar to that of Theorem 1. We remark that Theorem 2 remains true with quasi-ideal instead of bi-ideal. The proof is the same.

Theorem 3. *A semigroup S is a semilattice of groups if and only if the relation*

$$\bigcap_{i=1}^{k} Q_i = \prod_{i=1}^{k} Q_i \tag{6}$$

holds for any k quasi-ideals $Q_1, \cdots, Q_k$ of S (k is an arbitrary fixed positive integer greater than one).

It is easy to see that condition (1) holds in a semilattice of groups for left (and right) ideals too, instead of bi-ideals. The converse of this fact is also true in the following sense.

Theorem 4. *A semigroup S is a semilattice of groups if and only if condition* (1) *holds for any k left ideals and for any k right ideals of S (k is a fixed positive integer greater than one).*

The particular case of $k=2$ of Theorem 4 was formerly proved by the author [3]. Here we give a proof for $k=3$. Let S be a semigroup in which

$$A \cap B \cap C = ABC \tag{7}$$

holds for any triplet of left ideals and for any triplet of right ideals. Then (7) implies

$$L = SLS \tag{8}$$

and

$$R = SRS \tag{9}$$

for any left and right ideal of S, respectively. (8) and (9) imply that every one-sided (left or right) ideal of S is two-sided. To show that S is regular, let I be an arbitrary (two-sided) ideal of S. Then (7) implies (5) for any ideal I of S, which is equivalent with the regularity of S. Thus condition (7) is sufficient for S to be a semilattice of groups. The necessity follows from the above Corollary of Theorem A. The proof of the statement of Theorem 4 in case of $k>3$ is quite similar to that of the special case $k=3$.

Finally we announce a further criterion.

Theorem 5. *A semigroup S is a semilattice of groups if and only if S is centric and every principal left ideal of S is globally idempotent.*

References

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