# 9. *On H-closedness and the Wallman H-closed Extensions. II*[*)]

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**4. The Wallman H-closed extensions.** Let $X$ be a space, $\mathfrak{C}$ the family of all closed subsets of $X$, and $W(X)$ the collection of all subfamilies of $\mathfrak{C}$ which possess the PFIP and are maximal in $\mathfrak{C}$ relative to this property. Two elements $w_1, w_2$ of $W(X)$ are said to be equivalent if both of them contain the closures of the neighborhoods of the same point $x$ in $X$. An equivalent class in $W(X)$ corresponding to a point $x$ is called a fixed end and denoted by $\mathfrak{A}(x)$; an element in $W(X)$ which does not belong to any fixed end is called a free end and denoted by $\mathfrak{A}$. We denote by $\omega(X)$ the collection of all fixed and free ends in $X$. For an open subset $U$ of $X$ let $U^*=\{\mathfrak{A}(x)\,;\,x\in U\}$. We introduce the following topology for $\omega(X)$, called Katětov topology: the neighborhoods for fixed ends $\mathfrak{A}(x)$ are $U^*$ if $x\in U$ and for free ends $\mathfrak{A}$ are $U^*U\,\{\mathfrak{A}\}$, where $U$ is the interior of a closed set $A$ belonging to $\mathfrak{A}$. The space $\omega(X)$ with Katětov topology is $H$-closed and the subspace consisting of all $\mathfrak{A}(x)$ is homeomorphic to $X$ (also denoted by $X$). Moreover, the $H$-closed space $\omega(X)$ has the following properties: (1) $X$ is dense in $\omega(X)$, (2) $X$ is open in $\omega(X)$, and (3) $\omega(X)$-$X$ is discrete (see [5]).

**Lemma 5.** *Every bounded real-valued continuous function $f$ on $X$ can be continuously extended over $\omega(X)$.*

**Proof.** Suppose that $f$ can not be continuously extended at $\mathfrak{A}\in\omega(X)$. Then there is an $\varepsilon>0$ such that to the interior $U$ of each member $A$ of $\mathfrak{A}$ there are $x, y\in\bar{U}$ satisfying the condition $f(y)-f(x)>\varepsilon$. It is clear that for two members $A_\alpha, A_\beta$ of $\mathfrak{A}$ $f(y_\beta)-f(x_\alpha)>\varepsilon$, since $A_\alpha\cap A_\beta=A_{\alpha\beta}$, $f(y_{\alpha\beta})\le\min.\{f(y_\alpha), f(y_\beta)\}$, $f(x_{\alpha\beta})\ge\max.\{f(x_\alpha), f(x_\beta)\}$, and $f(y_{\alpha\beta})-f(x_{\alpha\beta})>\varepsilon$. Let $L$ be the least upper bound of $\{f(x_\alpha)\}$ and $M$ the greatest lower bound of $\{f(y_\alpha)\}$. Then $M>L$ and $M-L\ge\varepsilon$. If $P=\left\{x\,;\,f(x)\ge M-\frac{\varepsilon}{3}\right\}$ and $Q=\left\{x\,;\,f(x)\le L+\frac{\varepsilon}{3}\right\}$, then both $P$ and $Q$ intersect each member of $\mathfrak{A}$ in sets containing non-vacuous open sets and belong to $\mathfrak{A}$. But $P\cap Q=\emptyset$ and the contradiction proves the lemma.

**Corollary.** *Every unbounded real-valued continuous function on*

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*$X$ can be continuously extended to an extended continuous function over $\omega(X)$* (see [9] for proof).

It $C(\omega(X))$ is the algebra of all bounded real-valued continuous functions on $\omega(X)$, then $\omega(X)$ can be decomposed into disjoint closed subsets $S(x_0)=\{x\,;\ f(x)=f(x_0)$ for all $f\in C(\omega(X)),\ x, x_0\in\omega(X)\}$. A set of $S(x)$ is defined to be open if the union of the $S(x)$'s in the set is open in $X$. Then the mapping $\rho: x\rightarrow S(x)$ for $x\in\omega(X)$ is continuous and $\{S(x)\,;\ x\in\omega(X)\}$ form an $H$-closed space $\Omega(X)$.

**Theorem 6** (Stone-Čech). *If $X$ is a space separated by $C(X)$, the algebra of all bounded real-valued continuous functions on $X$ and $f$ is a continuous function on $X$ to an $H$-closed space $Y$, separated by $C(Y)$, then there is a pseudo-continuous extension of $f$ over $\Omega(X)$.*

**Proof.** Let $F(X)$ be the family of all continuous functions on $X$ to the closed unit interval $Q$ and $Q^F(X)$ the product of the unit interval $Q$ taken $F(X)$ times. Then $Q^{F(X)}$ is compact and the evaluation map $l$ carries an element $x$ of $X$ into the element $l(x)$ of $Q^{F(X)}$ whose $f$-th coordinate is $f(x)$ for each $f$ in $F(X)$. By Theorem 5 and Lemma 5, $\Omega(X)$ is pseudo-homeomorphic to $K(X)$ which is a closed subset of $Q^{F(X)}$, and $Y$ is pseudo-homeomorphic to $K(Y)\subset Q^{F(X)}$. A function $f^*$ on $F(Y)$ to $F(X)$ is induced by the given $f$ if we define $f^*(a)=a\circ f$ for each $a$ in $F(Y)$. Define $f^{**}$ on $Q^{F(X)}$ to $Q^{F(Y)}$ by letting $f^{**}(q)=q\circ f^*$ for each $q\in Q^{F(X)}$. Let $i$ be the embedding map of $X$ into $\Omega(X)$ and let $h$ and $g$ be evaluation map of $\Omega(X)$ and $Y$ into $K(\Omega(X))$ and $K(Y)$ respectively. Then $g^{-1}\circ f^{**}$ is the required pseudo-continuous extension of $f\circ h^{-1}\circ e^{-1}$.

**Theorem 7.** *A regular space $X$ is completely regular if $C(X)$ separates $X$.*

**Proof.** By Theorem 5 and Lemma 5, $\Omega(X)$ is pseudo-homorphic to compact space $K(X)\subset Q^{F(X)}$. Then $X$ is pseudo-homeomorphic to a subset of $K(X)$. The pseudo-homeomorphism between $X$ and the subset of $K(X)$ is, in fact, a homeomorphism on account of the regularity of the space $X$ [4, p. 43]. The theorem is proved.

**Lemma 6.** *If $A$ and $B$ are two open subsets of a completely regular space $X$, and $\bar{A}\cap\bar{B}=\emptyset$, then there is $f\in C(X)$, which takes the value 0 on $A$ and 1 on $B$.*

**Proof.** Let $\tilde{A}, \tilde{B}$ be the closures of $\bar{A}$ and $\bar{B}$ respectively in $\Omega(X)$ and $\tilde{C}(X)$ the extensions of the function in $C(X)$ over $\Omega(X)$. $\tilde{A}\cap\tilde{B}=\emptyset$. By Lemma 3, we can find an $\tilde{f}\in\tilde{C}(X)$ assuming 0 on $\tilde{A}$ and 1 on $\tilde{B}$. $f\in C(X)$ corresponding to $\tilde{f}$ is the required function.

Fan and Gottsman [3] showed that a regular space with a normal base can be embedded into a compact space, while all open sets in a completely regular space form a normal base by Lemma 6.

"**Theorem** (Fan and Gottsman). *A regular space is completely regular if and only if it has a normal base.*"

**5. The Stone-Weierstrass approximation theorem. Theorem 8** (Stone). *Let $R(X)$ be an algebra of real-valued continuous functions on an $H$-closed space $X$ containing constant functions and separating the points.*

Then every continuous function $f$ on $X$ is the limit of a uniformly convergent sequence of functions belonging to $R(X)$.

**Proof** (Stone). Every continuous function on an $H$-closed space is bounded and the uniform closure of $R(X)$ is a lattice. Let $\varepsilon > 0$ and $x_0, y_0 \in X$. There is a $g_{x_0 y_0}$ in $R(X)$ which satisfies the condition $g_{x_0 y_0}(x_0) = f(x)$ and $g_{x_0 y_0}(y_0) = f(y_0)$. We denote by $U_{x_0 y_0}, V_{x_0 y_0}$ the open sets in $X$ on which $g_{x_0 y_0}(x) < f(x) + \varepsilon$ and $g_{x_0} y_0(x) > f(x) - \varepsilon$. For fixed $y_0$, $\{U_{x_0 y_0}; x_0 \in X\}$ form an open cover of $X$. If $\{U_{x_1 y_0}, \cdots U_{x_n y_0}\}$ is a finite pseudo subcover of the open cover and we set $h_{y_0} = g_{x_1 y_0} \wedge \cdots \wedge g_{x_n} y_0$, then $h_{y_0}(x) \leq f(x) + \varepsilon$ for all $x$ and $h_{y_0}(x) > f(x) - \varepsilon$ on $V_{y_0} = \bigcap_{i=1}^{n} U_{x_i y_0}$. We can find a finite pseudo subcover $\{V_{y_1}, \cdots V_{y_m}\}$ of the open cover $\{V_{y_0}; y_0 \in X\}$. The function $p(x) = h_{y_1} \vee \cdots \vee h_{y_m}$ satisfies the inequalities $f(x) - \varepsilon \leq p(x) \leq f(x) + \varepsilon$ for all $x \in X$ and the proof is complete.

**Lemma 7.** *Let $\mathfrak{U}$ be an open cover of an $H$-closed space $X$ separated by $C(X)$. Then there exist a finite pseudo subcover $U_1, \cdots, U_n$ of $\mathfrak{U}$ and $n$ nonnegative real-valued continuous functions $f_1, \cdots, f_n$ on $X$ such that* (1) *$f_i$ vanish outside of $\bar{U}_i$ for $i = 1, \cdots, n$, and* (2) *$f_1(x) + \cdots + f_n(x) = 1$ for each $x \in X$.*

**Proof.** Each $x \in X$ belongs to some member $U$ of $\mathfrak{U}$. There is a nonnegative continuous function $g$ which vanishes outside of $\bar{U}$ and takes the value 1 at the point $x$ by Lemma 3. Let $V(x) = \{x; g(x) > 1/2\}$. Then $\{V(x): x \in X\}$ is an open cover of $X$ and has a finite pseudo subcover $\{V_1, \cdots, V_n\}$. Each $\bar{V}_i$ is contained in some $\bar{U}_i$ outside of which $g_i$ vanishes. Let $f_i = g_i/(g_1 + \cdots + g_n)$. Then $f_1 + \cdots + f_n = 1$ and the lemma is proved.

The set of the functions $f_1, \cdots, f_n$ in Lemma 5 is called a pseudo partition of unity associated with the open cover.

**Theorem 9** (Stone-Šilov-Weierstrass). *Let $X$ be a space separated by $C(X)$ and $\Omega(X)$ the Wallman $H$-closed extension of $X$ as before. If $S_0(X)$ is a self-adjoint subalgebra of the algebra $K(X)$ of all continuous complex-valued functions on $X$ and is contained in a closed subalgebra $S(X)$ of $K(X)$, then $f \in K(X)$ and $\tilde{f} \in \tilde{S}$ on every set of constancy for $S_0$ on $\Omega(X)$ imply that $f$ belongs to $S(X)$.*

For notations and the proof of the theorem see [10, p. 931] ("pseudo partition of unity" in Lemma 7 is used in lieu of "partition of unity").

Banaschewski [2] showed that each completely regular space $X$ has a non compact extension, a subset of $\omega(X)$, in which the Stone-Weierstrass theorem holds in the sense described in [1, Russian Math.

Surveys, p. 53] and Aleksandrov and Ponomarev raised the question whether the Stone-Weierstrass theorem holds in $\omega(X)$ [1, ibid, p. 54]. In order to solve the problem we first prove the following lemma.

**Lemma 8.** *For each completely regular space $X$ the continuous functions on $\omega(X)$ separate the points.*

**Proof.** It follows from Lemma 6 that a completely regular space $X$ has a normal base in Fan and Gottsman sense [3, p. 504] and thus, can be embedded in a compact space $X^*$. The free ends in $\omega(X)$ are the maximal binding families in $X$ [see 3 for definition]. By Lemma 5 each bounded continuous function on $X$ can be continuously extended over $X^*$ and each continuous function on $X^*$ is also continuous on $\omega(X)$.

**Theorem 10.** *For a completely regular space $X$ $\omega(X)=\Omega(X)$ and the Stone-Weierstrass theorem holds in $\omega(X)$.*

The first part of the theorem follows from Lemma 8 and the second part from Theorem 8.

**6. The Tietze extension theorem.** **Lemma 6.** *If $A$ and $B$ are two disjoint $H$-closed subsets of a space $X$ and $C(X)$ separates the points in $A\cup B$, then there is a continuous function $f$ on $X$ such that $f(A)=0$ and $f(B)=1$.*

**Theorem 11.** *If $A$ is an $H$-closed subset of a space $X$ and $C(X)$ separates the points in $A$, then each bounded continuous function $f$ on $A$ to $[-1, 1]$ can be continuously extended to $f$ over $X$ to $[-1, 1]$.*

**Proof** (Tietze). Let $C=\{x: f(x)\leq -1/3, x\in A\}$ and $D=\{x: f(x)\geq 1/3, x\in A\}$. Then $C$ and $D$ are disjoint $H$-closed sets and by Lemma 6′ there is $f_1$ on $X$ to $[-1/3, 1/3]$ such that $f_1(x)$ is $1/3$ on $C$ and $-1/3$ on $D$. $|f(x)-f_1(x)|\leq 2/3$ for all $x$ in $A$.

**Remark.** The condition that $C(X)$ separates the points in Theorems 8, 9, 11 is assumed for simplicity and more general results with slight modifications still hold without such restriction.

**7. Terminology.** The characterization of pseudo-compactness as the existence of a cluster point for each sequence of open sets was announced about the same time by (1) K. Iséki and S. Kasahara: Proc. Japan Acad., **33** (1957), (2) S. Mardésić and Z. P. Papić, Glasnik: Mat.-Fize. i Astr., **10** (1955), (3) J. D. McKnight, R. W. Bagley, and E. H. Connell: Bull. Amer. Math. Soc., **63**, 1 (1957), and (4) C. Wenjen: Bull. Amer. Math. Soc., **63**, 1 (1957), apparently under the influence of Hewitt's paper (Trans. Amer. Math. Soc., **64** (1948). The existence of a cluster point for each sequence of open sets and of a pseudo finite subcover for each countable open cover reveals the similarity between pseudo-compactness and countable compactness. On the other hand, the cluster point theorem for each net of open sets, the existence of a pseudo finite subcover for each open cover, and other properties of $H$-closed spaces (see Theorem 1, 3, 4) are just the analogues of the basic theorems for compact spaces. Even though pseudo-compactness has become a standard term, we feel strongly that the appropriate name for pseudo-compactness is "pseudo countable compactness" while *H*-closed spaces should be called "pseudo compact".

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