7. *Nonlinear Semi-groups in Banach Lattices*

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0. Introduction. Let X be a Banach lattice. X is not only a vector lattice, but also a real Banach space such that $|x|\leq|y|$ implies $\|x\|\leq\|y\|$. For elementary properties of vector lattices, we refer to K. Yosida [11].

R. S. Phillips [9], M. Hasegawa [5] and K. Sato [10] characterized the generators of strongly continuous non-negative (contraction) semi-groups of linear operators in Banach lattices, introducing their various notions of "dispersiveness".

In the present paper, we shall establish an extension of the above results on generation of semi-groups to the nonlinear cases, using a version of the estimates in M. G. Crandall-T. M. Liggett [3].

Developments of our approach to perturbation, convergence and approximation theories will be studied elsewhere.

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1. Dispersiveness. Following K. Sato [10], we define functionals $\tau: X\times X\to R$ by $\tau(x,y)=\lim_{\varepsilon\downarrow 0}(\|x+\varepsilon y\|-\|x\|)/\varepsilon$, and $\sigma: X^+\times X\to R$ by $\sigma(x,y)=\inf \tau(x,(y+k)\vee(-bx))$ for $x\in X^+$ where the infimum is taken for all $b\in R^+=[0,\infty)$ and all $k\in X$ satisfying $x\wedge|k|=0$, here X^+ denotes the cone of all non-negative elements of X.

Proposition 1 (K. Sato [10]). *Let $x\in X^+$, y and $z\in X$. Then*

(i) $-\|y^-\|\leq\sigma(x,y)\leq\|y^+\|$,

(ii) $\sigma(x,ay)=a\sigma(x,y)$ *for* $a\in R^+$,

(iii) $\sigma(x,ax+y)=a\|x\|+\sigma(x,y)$ *for* $a\in R$,

(iv) $\sigma(x,y+z)\leq\sigma(x,y)+\sigma(x,z)$,

(v) $y\leq z\Rightarrow\sigma(x,y)\leq\sigma(x,z)$,

(vi) $x\wedge|z|=0\Rightarrow\sigma(x,y)=\sigma(x,y+z)$.

As direct consequences of (i), (iv) *and* (vi), *we have*

(α) $\sigma(x,0)=0$, (β) $-\sigma(x,-y)\leq\sigma(x,y)$,

(γ) $|\sigma(x,y)-\sigma(x,z)|\leq\|y-z\|$, ($\delta$) $\sigma(0,y)=0$.

In the case of the Hilbert space $L^2(S,B,m)$,

$$\sigma(x,y)=(x,y)_{L^2}/\|x\|_{L^2} \quad \text{for} \quad x\in X^+\setminus\{0\}.$$

Many other examples have been discussed in [10].

Next, following R. S. Phillips [9], we can define a semi-inner-product [,] with the following properties:

(i) If $x \geq 0$, then $[y, x] \geq 0$ for all $y \geq 0$;

(ii) $[x, x^+] = \|x^+\|^2$.

Henceforth we fix one such semi-inner-product [,].

In what follows, a subset A of $X \times X$ shall be considered as a (generally multi-valued) operator of X into X.

We define as usual

(i) $Ax = \{y : (x, y) \in A\}$,

(ii) $D(A) = \{x : Ax \neq \emptyset\}$,

(iii) $R(A) = \bigcup_{x \in D(A)} Ax$.

If $A, B \subset X \times X$, and $\lambda \in R$, we set

(iv) $A + B = \{(x, y+z) : y \in Ax \text{ and } z \in Bx\}$,

(v) $\lambda A = \{(x, \lambda y) : y \in Ax\}$,

(vi) $A^{-1} = \{(y, x) : (x, y) \in A\}$.

If a subset A of $X \times X$ is single-valued, Ax will denote either the value of A at x or the set defined above in (i), depending on the context.

Definition (K. Sato [10]). Let $\omega \in R$ and A be a subset of $X \times X$. A is said to be

(i) *ω-dispersive in the strict sense* or *ω-dispersive* (*s*) if $\sigma((x_1 - x_2)^+, y_1 - y_2) \leq \omega \|(x_1 - x_2)^+\|$ for any $(x_i, y_i) \in A$, $i = 1, 2$;

(ii) *ω-dispersive in the wide sense* or *ω-dispersive* (*w*) if $\sigma((x_1 - x_2)^+, -(y_1 - y_2)) \geq -\omega \|(x_1 - x_2)^+\|$ for any $(x_i, y_i) \in A$, $i = 1, 2$;

(iii) *ω-dispersive with respect to* [,] if $[y_1 - y_2, (x_1 - x_2)^+] \leq \omega \|(x_1 - x_2)^+\|^2$ for any $(x_i, y_i) \in A, i = 1, 2$. The word "wide" in (ii) is justified by Proposition 1, (β).

Lemma 1. *Let A be ω-dispersive(w) or ω-dispersive with respect to* [,]. *Then $J_\lambda = (I - \lambda A)^{-1}$ is single-valued for any $\lambda \geq 0$ such that $\lambda\omega < 1$, and $\|(J_\lambda z_1 - J_\lambda z_2)^+\| \leq (1 - \lambda\omega)^{-1} \|(z_1 - z_2)^+\|$ for any $z_i \in R(I - \lambda A)$, $i = 1, 2$; here I is the diagonal of $X \times X$.*

Proof. We set $z_i = x_i - \lambda y_i$, $(x_i, y_i) \in A, i = 1, 2$. Let A be ω-dispersive(w). Using Proposition 1, we get the following inequality:

$$\begin{aligned}
\|(z_1 - z_2)^+\| &\geq \sigma((x_1 - x_2)^+, z_1 - z_2) \\
&= \sigma((x_1 - x_2)^+, x_1 - x_2 - \lambda(y_1 - y_2)) \\
&= \sigma((x_1 - x_2)^+, (x_1 - x_2)^+ - \lambda(y_1 - y_2)) \\
&= \|(x_1 - x_2)^+\| + \lambda\sigma((x_1 - x_2)^+, -(y_1 - y_2)) \\
&\geq \|(x_1 - x_2)^+\| - \lambda\omega \|(x_1 - x_2)^+\| = (1 - \lambda\omega) \|(x_1 - x_2)^+\|.
\end{aligned}$$

Next, let A be ω-dispersive with respect to [,]. Using the properties of [,], we get similarly to the above argument that

$$\begin{aligned}
\|(z_1 - z_2)^+\| \cdot \|(x_1 - x_2)^+\| &\geq [(z_1 - z_2)^+, (x_1 - x_2)^+] \\
&\geq [z_1 - z_2, (x_1 - x_2)^+] = [(x_1 - x_2) - \lambda(y_1 - y_2), (x_1 - x_2)^+] \\
&\geq \|(x_1 - x_2)^+\|^2 - \lambda\omega \|(x_1 - x_2)^+\|^2 = (1 - \lambda\omega) \|(x_1 - x_2)^+\|^2.
\end{aligned}$$

Each inequality shows that J_λ is single-valued and

$$\|(J_\lambda z_1 - J_\lambda z_2)^+\| \leq (1 - \lambda\omega)^{-1} \|(z_1 - z_2)^+\|. \qquad \text{Q.E.D.}$$

Remark 1. If A is ω-dispersive(w) or ω-dispersive with respect to [,] and is a single-valued linear operator satisfying $R(I-\lambda A)=X$ for sufficiently small $\lambda>0$, then $A-\omega I$ is dissipative.

Let X be an abstract L^1-space introduced by S. Kakutani (see, for example, [11] p. 369). If A is ω-dispersive(w) or ω-dispersive with respect to [,], then $A-\omega I$ is dissipative in the sense of [4], [8].

In general, however, we may only conclude that

$$\|J_\lambda z_1 - J_\lambda z_2\| \leq 2(1-\lambda\omega)^{-1}\|z_1 - z_2\| \text{ for } z_i \in R(I-\lambda A), \\ i=1,2;\ \lambda \geq 0,\ \lambda\omega < 1.$$

Nevertheless, Lemma 1 enables us to take the argument parallel to that used in M. G. Crandall-T. M. Liggett [3], thanks to the following lemma.

Lemma 2. *Let* $d_\pm(x,y)=\|(x-y)^\pm\|$. *Then*

(i) $d_\pm(x,x)=0$,

(ii) $d_\pm(x+z, y+z)=d_\pm(x,y)$,

(iii) $d_\pm(\alpha x, \alpha y)=\alpha d_\pm(x,y)$ for $\alpha \in R^+$,

(iv) $d_\pm(x,z) \leq d_\pm(x,y)+d_\pm(y,z)$,

(v) $\|x-y\| \leq d_+(x,y)+d_-(x,y)$.

Proof. (i) and (ii) are evident. (iii), (iv) and (v) are respectively consequences of the following facts:

(iii′) $(\alpha x-\alpha y)^\pm=\alpha(x-y)^\pm$,

(iv′) $|(x-z)^\pm| \leq |(x-y)^\pm|+|(y-z)^\pm|$,

(v′) $|x-y|=(x-y)^+ + |(x-y)^-|$. Q.E.D.

2. Generation of semi-groups. If C is a subset of X, a semi-group S on C is a function on R^+ such that $S(t)$ maps C into C for each $t \in R^+$ and S satisfies $S(t+\tau)=S(t)S(\tau)$ for all $t, \tau \in R^+$ and $\lim_{t \downarrow 0} S(t)x=S(0)x=x$ for any $x \in C$.

Definition. A semi-group S on C is called *an order-preserving semi-group of type* ω if it satisfies

$$\|(S(t)x-S(t)y)^+\| \leq e^{\omega t}\|(x-y)^+\| \qquad (*)$$

for all $t \in R^+$ and $x, y \in C$, and we denote by $Q_\omega^+(C)$ the totality of such semi-groups.

Remark 2. From $(*)$ we have that $x \leq y$ implies $S(t)x \leq S(t)y$ for all $t \in R^+$. But $e^{-\omega t}S(t)$ seems in general not to be a contraction. See also Remark 1.

Theorem A. (i) *Let* A_0 *be the strict infinitesimal generator of* $S \in Q_\omega^+(C)$. *Then* A_0 *is* ω*-dispersive(s).*

(ii) *Let* $\Phi=\{\varphi\}$ *be an ultra-filter of sets* $\varphi \subset (0,\infty)$, *which converges to* 0, *and let* A_Φ *be the* Φ*-infinitesimal generator of* $S \in Q_\omega^+(C)$. *Then* A_Φ *is* ω*-dispersive with respect to* [,]. (For the definitions of A_0 and A_Φ, we refer to Y. Kōmura [7].)

Proof. (i) Let $x_i \in D(A_0)$, $i=1,2$. Using Proposition 1, we obtain that for $h>0$,

$$\begin{aligned}&\sigma((x_1-x_2)^+, S(h)x_1-x_1-(S(h)x_2-x_2))\\&=\sigma((x_1-x_2)^+, S(h)x_1-S(h)x_2-(x_1-x_2)^+)\\&=-\|(x_1-x_2)^+\|+\sigma((x_1-x_2)^+, S(h)x_1-S(h)x_2)\\&\leq -\|(x_1-x_2)^+\|+\|S(h)x_1-S(h)x_2)^+\|\\&\leq (e^{\omega h}-1)\|(x_1-x_2)^+\|.\end{aligned}$$

Multiply this by h^{-1}, use (ii) of Proposition 1, and make h tend to zero. Then we get

$$\sigma((x_1-x_2)^+, A_0x_1-A_0x_2)\leq \omega\|(x_1-x_2)^+\|.$$

(ii) Let $x_i \in D(A_\phi)$, $i=1,2$. Using the properties of [,], we get similarly to the above argument that for $h>0$,

$$\begin{aligned}&[S(h)x_1-x_1-(S(h)x_2-x_2), (x_1-x_2)^+]\\&=[S(h)x_1-S(h)x_2, (x_1-x_2)^+]-\|(x_1-x_2)^+\|^2\\&\leq [(S(h)x_1-S(h)x_2)^+, (x_1-x_2)^+]-\|(x_1-x_2)^+\|^2\\&\leq \|(S(h)x_1-S(h)x_2)^+\|\cdot\|(x_1-x_2)^+\|-\|(x_1-x_2)^+\|^2\\&\leq (e^{\omega h}-1)\|(x_1-x_2)^+\|^2.\end{aligned}$$

Therefore $[A_\phi x_1-A_\phi x_2, (x_1-x_2)^+]\leq \omega\|(x_1-x_2)^+\|^2$. Q.E.D.

Theorem B. *Let $A\subset X\times X$ be ω-dispersive(w) or ω-dispersive with respect to [,] and assume that $R(I-\lambda A)\supset\overline{D(A)}$ for all sufficiently small positive λ. Then*

$$S(t)x=\lim_{n\to\infty}\{I-(t/n)A\}^{-n}x \qquad (**)$$

exists for any $x\in\overline{D(A)}$, $t\in R^+$. Moreover $S\in Q^+_\omega(\overline{D(A)})$.

Proof. By Lemma 1 and Lemma 2, we can take the argument parallel to that used in the proof of Theorem I in [3] and obtain, for example, the following estimates: For any $x\in D(A)$,

(i) $d_\pm(J^n_{t/n}x, J^m_{t/m}x)\leq 2t\ \exp\ (4|\omega|t)(1/m-1/n)^{1/2}|||Ax|||$ for $n\geq m$, $t\in R^+$, where $|||Ax|||=\inf\{\|y\|: y\in Ax\}$ ((1.10) in [3]),

(ii) $d_\pm(S(\tau)x, S(t)x)\leq\{\exp\,(2|\omega|(t+\tau))+\exp\,(4\,|\omega|t)\}\,|||Ax|||\cdot(\tau-t)$ for $\tau>t\geq 0$ ((1.11) in [3]).

In this way we may derive the conclusion. Q.E.D.

3. Example. Let Ω be a bounded domain in R^n with smooth boundary and let $\beta\subset R\times R$ be a maximal monotone set such that $0\in D(\beta)$. We introduce the following operator:

$$\bar\beta=\{(u, v)\in L^2(\Omega)\times L^2(\Omega): v(\omega)\in\beta(u(\omega)) \text{ a.e. in } \Omega\}.$$

Proposition 2 (H. Brezis-M. G. Crandall-A. Pazy [1]). *The operator A defined by $Au=\Delta u-\bar\beta(u)$ with the domain $D(A)=H^2(\Omega)\cap H^1_0(\Omega)\cap D(\bar\beta)$ is dissipative and satisfies $R(I-A)=L^2(\Omega)$.*

By the well-known theorems (in [4], [7]) for nonlinear contraction semi-groups in Hilbert spaces, A generates a nonlinear contraction semi-group S on $\overline{D(A)}$, which is easily seen to admit of the representation (**) according to, for instance, [2] and [8].

Lemma 3. *A is* 0*-dispersive(s) and also* 0*-dispersive with respect to* [,].

Proof. The operator Δ with domain $H^2(\Omega)\cap H_0^1(\Omega)$ is 0-dispersive(s) and 0-dispersive with respect to [,] since it is a generator of a non-negative contraction semi-group of linear operators. On the other hand, $-\bar{\beta}$ is 0-dispersive(s) and also 0-dispersive with respect to [,] since $[x, y]=(x, y)_{L^2}$ and $\|x\|_{L^2}\cdot\sigma(x, y)=(x, y)_{L^2}$. Hence A also has the dispersivities of the same type. Q.E.D.

From this lemma and Theorem B, we obtain

Theorem C. *The operator A in Proposition* 2 *generates an order-preserving semi-group of type* 0 *of contractions on* $\overline{D(A)}\subset L^2(\Omega)$.

In particular, if u_1 and u_2 are solutions of the equation $\partial u/\partial t-\Delta u+u^{2p+1}=0$ (p is any non-negative integer) in $L^2(\Omega)$ and if $u_1(0, \omega)\geq u_2(0, \omega)$ for a.e. $\omega\in\Omega$, then $u_1(t, \omega)\geq u_2(t, \omega)$ for a.e. $\omega\in\Omega$ for all $t\in R^+$.

Added in proof. After this paper was submitted for publication, the author became aware of [12].

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