6. *Construction of a Local Elementary Solution for Linear Partial Differential Operators. I*

By Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University

(Comm. by Kunihiko KODAIRA, M. J. A., Jan. 12, 1971)

Let $P(x, D_x)$ be a partial differential operator with real analytic coefficients. Assume that the principal part P_m of P is simple characteristic and that P_m is of real coefficients. The purpose of this note is to construct $E(x, y)$ which satisfies $P(x, D_x)E(x, y)=\delta(x-y)$ near (x_0, x_0, ξ_0) as sections of the sheaf $\mathcal{C}$, where ξ_0 is a cotangent vector at x_0. (We refer the reader to Sato [7], [8] about the notion of the sheaf $\mathcal{C}$ defined on the cotangential sphere (or co-sphere) bundle. See also Kashiwara and Kawai [3]). In other words we construct a local co-spherical elementary solution for $P(x, D_x)$. We construct $E(x, y)$ in two different methods. The first one relies on the analysis in a complex domain and the second on the theory of pseudo-differential operators of finite type developed in Kashiwara and Kawai [3]. The extension of our theory to the operators with complex coefficients will be given in our forthcoming note. The details of this note will be published elsewhere. (See also Kawai [5].)

1°. We begin with the following Theorem 1 essentially due to Hamada [1], which treats the singular Cauchy problem in a complex domain.

Let $P(z, D_z)$ be a linear partial differential operator with holomorphic coefficients defined near the origin of C^n and have the form $P(z, D_z)=\sum_{j=0}^m a_j(z, D_{z'})\partial^j/\partial z_1^j$, where $a_0(z, D_z)\equiv 1$, $z'=(z_2, \cdots, z_n)$ and $a_j(z, D_{z'})$ is a differential operator of order at most $(m-j)$.

Denote by $P_m(z, \xi)$ the principal symbol of $P(z, D_z)$ where ξ is a cotangent vector at z which stands for D_z. Assume further that one of the solutions $\xi_1^0(z\,;\xi_2, \cdots, \xi_n)$ of $P_m(z\,;\xi_1, \xi_2, \cdots, \xi_n)=0$ is holomorphic in (z, ξ') and that $\partial/\partial\xi_1(P_m(z\,;\xi_1^0, \xi_2, \cdots, \xi_n))\neq 0$ near $(z, \xi')=(0, \xi_0')$, where $\xi'=(\xi_2, \cdots, \xi_n)$.

We denote by $\varphi(z, \xi'\,; s, y')$ the phase function with parameter $(s, y')=(s, y_2, \cdots, y_n)$ corresponding to ξ_1^0, that is, the characteristic function of $P(z, D_z)$ satisfying

(i) $P_m(z, \mathrm{grad}_z\varphi)\equiv 0$

(ii) $\varphi(s, z', \xi', s, y')=\langle z'-y', \xi'\rangle$

(iii) $\mathrm{grad}_z\varphi|_{z_1=s}=(\xi_1^0(s, z'\,;\xi_2, \cdots, \xi_n), \xi_2, \cdots, \xi_n)$.

Then we have

Theorem 1. *The following singular Cauchy problem* (SC) *has a local solution* $u(z,\xi';s,y')$ *which is multivalued analytic except on* $K(\xi',s,y')=\{z\,|\,\varphi(z,\xi';s,y')=0\}$.

$$(\mathrm{SC})\quad \begin{cases} P(z,D_z)u(z,\xi';s,y')=0 \\ P'(z,D_z)u(z,\xi';s,y')|_{z_1=s}=1/(\langle z'-y',\xi'\rangle)^n, \end{cases}$$

where

$$P'(z,D_z)=\sum_{j=1}^{m} a_j(z,D_{z'})\frac{\partial^{j-1}}{\partial z_1^{j-1}}.$$

Here the existence domain of u *can be taken independent of* (ξ',s,y') $(|\xi'-\xi_0'|,|s|,|y'|\ll 1)$.

Since $P'_{m-1}(z,\mathrm{grad}_z\varphi)=(\partial/\partial\xi_1)P_m(z;\xi)|_{\xi=\mathrm{grad}_z\varphi}\neq 0$ by the assumption, we can construct $u(z,\xi';s,y')$ in a cannonical way using the method of asymptotic expansions just as in Hamada [1] (see also Kawai [4] Theorem 2). We remark here that the above condition on P is weaker than the usual assumption of simple characteristics, that is, the condition on P is, so to speak, the assumption of directional simple characteristics. We can also weaken the above condition to the assumption of constant multiple characteristics in some cases, but we do not treat the case in this note. (In that case the solution $u(z,\xi';s,y')$ has essential singularities and the logarithmic singularities on $K(\xi',s,y')$ in general.)

Now we return to the analysis in the real domain and obtain the following

Theorem 2. *Let* $P(x,D_x)$ *be a partial differential operator with analytic coefficients. Assume further that the principal part* $P_m(x,D_x)$ *of* $P(x,D_x)$ *has real coefficients and is simple characteristic. Then we can find locally a hyperfunction* $E(x,y;\xi)$ *which satisfies* $P(x,D_x)E(x,y;\xi)=1/(\langle x-y,\xi\rangle+i0)^n$ *and depends real analytically on* y *and* ξ.

Remark. The assumption that P_m has simple characteristics is redundunt. It is clear from the method of our proof that we need only the assumption of directional simple characteristics near ξ_0.

Sketch of the proof. We can assume without the loss of generalities that $P(z,D_z)$ satisfies the conditions of Theorem 1 by the assumption of simple characteristics. If $\xi=(1,0,\cdots,0)$, then $P_m(x_0,\xi)\neq 0$ by the above assumption and it is already known that we can construct $E(x,y;\xi)$ with the required properties (Sato [6], see also Kashiwara and Kawai [3]). Therefore we assume that $\xi=(\xi_1,\cdots,\xi_n)$ is not parallel to $(1,0,\cdots,0)$. So we can consider $\xi_2\neq 0$ without the loss of generalities. To construct $E(x,y;\xi)$ with the required properties, it is sufficient to construct $E(x,y;\xi)$ when $y=0$ and $\xi_1=0$ because real coordinate transformation $\{v_j=x_j-y_i\,(j\neq 2),\, v_2=\xi_1(x_1-y_1)/\xi_2+(x_2-y_2)\}$ reduces the problem to that case. For the sake of simplicity we use x

instead of v even after the above transformation. By Theorem 1 there exists a function $u(z, \xi'; s)$ which satisfies

$$\begin{cases} P(z, D_z)u(z, \xi'; s)=0 \\ P'(z, D_z)u(z, \xi'; s)|_{z_1=s}=1/\langle z', \xi'\rangle^n, \end{cases}$$

where s is a real number and $|s|\ll 1$. Assuming $\operatorname{Im}\varphi(x_1, z', \xi'; s)>0$ and x_1 is real we define $E(x_1, z'; \xi')=\int_0^{x_1} u(x_1, z', \xi'; s)ds$. By the definition of the operator $P'(z,D_z)$ we conclude that $P(x_1, z', \partial/\partial x_1, \partial/\partial z_2, \cdots, \partial/\partial z_n)E(x_1, z'; \xi')=1/\langle z', \xi'\rangle^n$. Since we have assumed that the principal part of P has real coefficients, we can suppose the phase function φ is a real valued function. Therefore the uniform analytic function $E(x_1, z'; \xi')$ defines a hyperfunction $E(x; \xi')$ as its boundary value from the domain $\{\operatorname{Im}\varphi>0\}$. Thus we have obtained the required $E(x, y; \xi)$.

Remark 1. Taking a neighbourhood U of ξ_0 in $(n-1)$-dimensional sphere we define $E(x, y)$ by

$$\frac{(n-1)!}{(-2\pi i)^n}\int_{\xi\in U} E(x, y, \xi) \sum_{j=1}^{n} (-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n.$$

Then the above method of the construction of E and the theory of integration on the sheaf $\mathcal{C}$ (see Sato [8]) show that the singular support of $E(x, y)$ is contained in the union of the sets of the type $\{(x, x, \xi, -\xi)\}$ $\cup\{(x, y, \xi, -\eta)\,|\,(y, \eta)$ belongs to the bicharacteristic strip of $P(x, D_x)$ issuing from (x, ξ) and $x_1>y_1\}$ near $(x_0, x_0, \xi_0, -\xi_0)$. Thus Theorem 6 of Kawai [4] is the best possible one.

Remark 2. Since we want to treat the problem on the co-sphere bundle, we have presented the theorem localizing the statement in ξ-space. But, as we have assumed that the principal part of P has real coefficients and is of simple characteristics, we can also construct $E(x, y)$ satisfying $P(x, D_x)E(x, y)=\delta(x-y)$ not localizing the statement with respect to ξ. In fact we consider in that case a singular Cauchy problem in a complex domain giving the Cauchy data by $(u(z_1, z'), \partial/\partial z_1 u(z_1, z'), \cdots, \partial^{m-1}/\partial z_1^{m-1} u(z_1, z'))|_{z_1=s}=(0, \cdots, 0, 1/\langle z'-y', \xi'\rangle^n)$, instead of (SC) in Theorem 1. Since the existence and the uniqueness of the solution for the above Cauchy problem has been proved by Hamada [1] (see also Kawai [4] Theorem 1), we can construct $E(x, y)$ just in the same way as above.

2^0. In this paragraph we give, as an application of our previous note (Kashiwara and Kawai [3]), another method of constructing a local elementary solution of a partial differential operator $P(x, D_x)$ with simple characteristics and real coefficients in its principal part, following the idea of Hörmander (Hörmander [2]; there he uses the Fourier transform technique rather than the Radon transform technique, which we employ in the sequel). The author expresses his sincere thanks to Professor Gårding, who kindly called the author's attention to Hörmander's idea. Professor Kotake also kindly suggested the author by correspondence that he should employ Hörmander's idea. We hope the use of the theory of the sheaf C and the assumption that the coefficients of P are analytic have made the situation transparent. We also remark that the singular support in the theory of hyperfunctions is defined *modulo real analytic functions*, not modulo C^∞ functions.

Theorem 2′. *Let P be as in Theorem 2. Then we can construct locally $F(x, y, \xi)$ such that $P(x, D_x)\int_U F(x, y, \xi)\omega(\xi)$ defines a kernel function of some elliptic pseudo-differential operator of finite type near (x_0, ξ_0), where $\omega(\xi)=\sum_{j=1}^n (-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$ and U is a neighbourhood of ξ_0 on the $(n-1)$-dimensional sphere.*

See Kashiwara and Kawai [3] about the notion of (elliptic) pseudo-differential operators of finite type.

Corollary. *Since the elliptic pseudo-differential operator of finite type is invertible (Kashiwara and Kawai [3] Theorem 6), there exists $E(x, y)$ such that $P(x, D_x)E(x, y)=\delta(x-y)$ near $(x_0, x_0, \xi_0, -\xi_0)$.*

Sketch of the proof of Theorem 2′. We first choose a real phase function $\varphi(x, y, \xi)$ which is positively homogeneous of degree 1 with respect to ξ satisfying $\varphi(x, y, \xi)=\langle x-y, \xi\rangle+O(|x-y|^2|\xi|)(|x-y|\ll 1)$ and $P_m(x, \mathrm{grad}_x\varphi(x, y, \xi))\equiv P_m(y, \xi)$. We want to find $F(x, y, \xi)$ in the form of $\sum_{j\geq 0} f_j(x, y, \xi)\Phi_j(\varphi(x, y, \xi)+i0)/(P_m(y, \xi)+i0)$, where

$$\Phi_j(\tau)=\begin{cases}\dfrac{(-1)^j(n-j-1)!}{(-2\pi\sqrt{-1})^n}\dfrac{1}{\tau^{n-j}} & (j<n)\\[2ex] \dfrac{-1}{(2\pi\sqrt{-1})^n(j-n)!}\left\{\tau^{j-n}\log\tau-\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{j-n}\right)\tau^{j-n}\right\} & (j\geqq n)\end{cases}$$

and $f_j(x, y, \xi)$ is real analytic in $(x, y, \xi)(\xi\neq 0)$ and positively homogeneous of degree $(-j)$ with respect to ξ. Here the symbol $+i0$ means the fact that $F(x, y, \xi)$ is defined as the boundary value from the domain $\{\mathrm{Im}\,\varphi(z, w, \zeta)>0$ and $\mathrm{Im}\,P_m(w, \zeta)>0\}$ of some holomorphic function $\sum_{j\geq 0} f_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))/P_m(w, \zeta)$ in (z, w, ζ). That is, we want to find some holomorphic function $F(z, w, \zeta)$ such that $(*)\, P(z, \partial/\partial z)F(z, w, \zeta)=\sum_{j\geq 0} r_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))$ holds with $r_0(z, w, \zeta)\neq 0$ in a complex neighbourhood of (x_0, x_0, ξ_0). Substituting $\sum_{j\geq 0} f_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))(1/P_m(w, \zeta))$ for $F(z, w, \zeta)$ we can determine $f_j(z, w, \zeta)$ successively by solving the first order partial differential equation $\sum_{k=1}^n a_k\, \partial f_j/\partial z_k+bf_j=\rho_j$, where $a_k=\partial/\partial\xi_k\, P_m(z, \xi)|_{\xi=\mathrm{grad}_z\varphi}$, $b=P_{m-1}(z, \mathrm{grad}_z\varphi)+\cdots$, $\rho_0\equiv 0$ and $\rho_j(j\geqq 1)$ is determined by $\{f_0, \cdots, f_{j-1}\}$. Since we have assumed that the operator P is of simple characteristics, we can find some non-characteristic surface S for the above first order equation. Giving the Cauchy data on S to the above equation by 1 for $j=0$ and by 0 for $j\geqq 1$, we have the inequality $\sup_{(z,w,\zeta)\in V}|f_j(z,w,\zeta)|\leqq C^j j!\ (j\gg 0)$ for some complex neighbourhood V of (x_0, x_0, ξ_0) and for some constant C. By this estimate the above relation $(*)$ holds as an equality for holomorphic functions in $\{(z,w,\zeta)\in V \mid \operatorname{Im}\varphi(z,w,\zeta)>0, \operatorname{Im}P_m(w,\zeta)>0, \text{ and } |\varphi|<1/2C\}$. Since the Cauchy data for $f_0=r_0$ is 1, r_0 is not equal to zero in some complex neighbourhood of (x_0, x_0, ξ_0). Thus we have determind $F(z,w,\zeta)$ with the properties required above. By the assumptions on the operator P and the phase function φ it is easy to verify that $F(z,w,\zeta)$ defines a hyperfunction $F(x,y,\xi)$ if we choose ζ for which $\langle \operatorname{grad} P_m(x, \operatorname{grad}\varphi(x,y,\xi)), \operatorname{Im}\zeta\rangle>0$ holds. We can also verify that Remark 1 of Theorem 2 holds for $\int F(x,y,\xi)\omega(\xi)$. It is obvious that $P(x,D_x)\int F(x,y,\xi)\omega(\xi)$ defines an elliptic pseudo-differential operator of finite type with its principal symbol equal to $r_0\times(\partial(\psi_1,\cdots,\psi_n)/\partial(\xi_1,\cdots,\xi_n))^{-1}$ where ψ_j's are chosen so that they are real valued, positively homogeneous of degree 1 with respect to ξ and satisfy $\varphi=\sum_{j=1}^n(x_j-y_j)\psi_j$. (It is proved by Sato in much more general situations that the ambiguity of the choice of ψ_j's has no effect after the integration). Therefore $F(x,y,\xi)$ has all properties we wanted.

References

[1] Hamada, Y.: The singularities of the solutions of the Cauchy problem. Publ. R.I.M.S. Kyoto Univ. Ser. A, **5**, 21–40 (1969).

[2] Hörmander, L.: On the singularity of solutions of partial differential equations. Proc. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, pp. 31–40 (1969).

[3] Kashiwara, M., and T. Kawai: Pseudo-differential operators in the theory of hyperfunctions. Proc. Japan Acad., **46**(10), 1130–1134 (1970).

[4] Kawai, T.: Construction of elementary solutions for I-hyperbolic operators and solutions with small singularities. Proc. Japan Acad., **46**, 912–916 (1970).

[5] ——: On the local theory of (pseudo)-differential operators. Reports of the Symposium on the Theory of Hyperfunctions, R.I.M.S. Kyoto University, pp. 1–45 (1970) (in Japanese).

[6] Sato, M.: Hyperfunctions and partial differential equations. Proc. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, pp. 91–94 (1969).

[7] ——: Hyperfunctions and Partial Differential Equations. Proc. of Nice Congress (1970).

[8] ——: On the structure of hyperfunctions. Sûgaku no Ayumi, **15**, 9–72 (1970) (Notes by Kashiwara, in Japanese).