5. *A Remark on the Meet Decomposition of Ideals in Noncommutative Rings*[*)]

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Introduction. In his paper [4] N. Radu has called that a commutative ring R is in the class $\mathfrak{D}$ if every ideal of R is represented as an intersection of primary ideals of R, and has shown that if R is in the class $\mathfrak{D}$, then $CB+A=C+A$ holds for ideals A, B and C of R such that $C\subseteq \bigcap_{\alpha\in I_B}(A+B_\alpha)$, where $\{B_\alpha \mid \alpha\in I_B\}$ is the set of all ideals which have the same nilradical with that of B.

The aim of this note is to generalize the above fact to noncommutative rings. Throughout this note, R is a noncommutative ring. The existence of unity is not assumed. The term *ideals* mean *two-sided ideals*, and (x) means the principal ideal generated by an element x. An ideal Q of R is called a (*right*) *M-primary* [n-primary] ideal if $AB\subseteq Q$ and $A\not\subseteq Q$, for ideals A and B, imply that B is contained in the McCoy's [nilpotent] radical of Q. The *right residual* of an ideal A by an ideal B is denoted by $A:B$, that is, $A:B=\{x\in R \mid xB\subseteq A\}$. A ring R will be called that it is in *the class* $\mathfrak{D}$ *with respect to the McCoy's* [*nilpotent*] *radical* if every ideal of R is represented as an intersection of M-primary [n-primary] ideals of R.

§1. Throughout this note, $\bar{A}$ will denote *the McCoy's radical* of an ideal A of R, that is, $\bar{A}$ is the intersection of all minimal prime ideals containing A. For an ideal B, I_B will mean the set of the indices of the ideals B_α with $\bar{B}_\alpha=\bar{B}$.

Lemma 1. *The following conditions are equivalent*:

(1) *R is in the class* $\mathfrak{D}$ *with respect to the McCoy's* [*nilpotent*] *radical.*

(2) *Every strongly meet irreducible ideal is M-primary* [*n-primary*].

Proof. This is immediate from the fact that every ideal is represented as an intersection of strongly meet irreducible ideals.

Theorem 1. *The following conditions are equivalent*:

(1) *R is in the class* $\mathfrak{D}$ *with respect to the McCoy's radical.*

(2) *If A, B and C are ideals such that* $C\subseteq \bigcap_{\alpha\in I_B}(A+B_\alpha)$ *then* $CB+A=C+A$.

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*) Dedicated to Professor K. Asano on his sixtieth birthday.

(3) $A=\bigcap_{\alpha\in I_B}(A+B_\alpha)\cap(A:B)$ *for any ideals* A *and* B *of* R.

(4) $A=\bigcap_{\alpha\in I_B}(A+B)\cap(\bigcup_{\alpha\in I_B}(A:B))$ *for any ideals* A *and* B *of* R.

Proof. By using (2) in Lemma 1, we can prove the theorem by the following implications: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1).

(1)$\Rightarrow$(2): Evidently we have $CB+A\subseteq C+A$. Conversely, we shall show that if any M-primary ideal Q contains $CB+A$, then Q contains $C+A$. Now suppose that $Q\supseteq CB+A$, then we have $Q\supseteq CB$ and $Q\supseteq A$. Moreover if $\bar{Q}\not\supseteq B$, then $Q\supseteq C$. If $\bar{Q}\supseteq B$, then there exists $\alpha_0\in I_B$ such that $Q\supseteq B_{\alpha_0}$. For, set $B\cap Q=B_{\alpha_0}$, then $\bar{B}_{\alpha_0}=\bar{B}\cap\bar{Q}=\bar{B}$. Hence we have that $C\subseteq\bigcap_{\alpha\in I_B}(A+B_\alpha)\subseteq A+B_{\alpha_0}\subseteq Q$. Therefore in both cases we have $Q\supseteq C+A$.

(2)$\Rightarrow$(3): If there exist two ideals A and B such that $A\subsetneqq\bigcap_{\alpha\in I_B}(A+B_\alpha)\cap(A:B)$, then we can find an element x in $\bigcap_{\alpha\in I_B}(A+B_\alpha)\cap(A:B)$ but not in A. Then x is in $\bigcap_{\alpha\in I_B}(A+B_\alpha)$. Hence we have $(x)+A=(x)B+A$. On the other hand, we obtain $(x)B\subseteq A$ for $x\in A:B$. Hence we have $(x)+A=A$. This implies that $x\in A$, which is a contradiction.

(3)$\Rightarrow$(4): For every $B_\beta\in\{B_\alpha\mid\alpha\in I_B\}$, we have $A=\bigcap_{\alpha\in I_B}(A+B_\alpha)\cap(A:B_\beta)$. Hence we have $A=\bigcup_{\beta\in I_B}(\bigcap_{\alpha\in I_B}(A+B_\alpha)\cap(A:B_\beta))=(\bigcap_{\alpha\in I_B}(A+B)\cap(\bigcup_{\beta\in I_B}(A:B_\beta))$.

(4)$\Rightarrow$(1): If there exists a strongly meet irreducible ideal Q which is not M-primary, then we have two ideals A and B such that $AB\subseteq Q$, $A\not\subseteq Q$ and $B\not\subseteq\bar{Q}$. Hence we have $Q:B\supsetneqq Q$. Now we shall prove that no $B_\alpha\in\{B_\alpha\mid\alpha\in I_B\}$ is contained in Q. If there exists B_α such that $B_\alpha\subseteq Q$, then $\bar{B}_\alpha\subseteq\bar{Q}$. Since $\bar{B}_\alpha=\bar{B}$, we have $\bar{B}\subseteq\bar{Q}$. This implies $B\subseteq\bar{Q}$, which is a contradiction. Therefore we obtain $Q+B_\alpha\supsetneqq Q$ for every $\alpha\in I_B$. Hence we have $Q\subsetneqq\bigcap_{\alpha\in I_B}(Q+B_\alpha)\cap(Q:B)\subseteq\bigcap_{\alpha\in I_B}(Q+B_\alpha)\cap(\bigcup_{\alpha\in I_B}(Q:B_\alpha))$.

§2. We let $\tilde{A}$ be *the nilpotent radical* of an ideal A of R, that is, $\tilde{A}=\{x\in R\mid(x)^k\subseteq Q$ for some positive integer $k\}$.

Theorem 2. *The following conditions are equivalent*:

(1) *R is in the class $\mathfrak{D}$ with respect to the nilpotent radical.*

(2) *If A, N and C are ideals such that $C\subseteq\bigcap_{n=1}^{\infty}(A+N^n)$ and N is a finitely generated ideal, then $CN+A=C+A$.*

(3) *If A, (b) and C are ideals such that $C\subseteq\bigcap_{n=1}^{\infty}(A+(b)^n)$, then $C(b)+A=C+A$.*

(4) $A=\bigcap_{n=1}^{\infty}(A+N^n)\cap(A:N)$ *for any ideal A and any finitely generated ideal N.*

(5) $A=\bigcap_{n=1}^{\infty}(A+(b)^n)\cap(A:(b))$ *for any ideal* A *and any element* b.

Proof. By using Lemma 1, we can prove the theorem by the following implications: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(5)$\Rightarrow$(1) and (2)$\Rightarrow$(4)$\Rightarrow$(5).

(1)$\Rightarrow$(2): Evidently we have $CN+A\subseteq C+A$. Conversely, we shall show that if any n-primary ideal Q contains $CN+A$, then Q contains $C+A$. Now suppose that $Q\supseteq CN+A$, then we have $Q\supseteq CN$ and $Q\supseteq A$. Moreover if $\tilde{Q}\not\supseteq N$, then $Q\supseteq C$. If $\tilde{Q}\supseteq N$, then there exists a positive integer k such that $N^k\subseteq Q$. Hence we have $C\subseteq\bigcap_{n=1}^{\infty}(A+N^n)\subseteq A+N^k\subseteq Q$. Therefore in both cases we have $Q\supseteq C+A$.

(2)$\Rightarrow$(3) and (4)$\Rightarrow$(5): These are immediate.

(3)$\Rightarrow$(5) and (2)$\Rightarrow$(4): These are similar to the proof of Theorem 1.

(5)$\Rightarrow$(1): If there exists a strongly meet irreducible ideal Q which is not n-primary, then we can find two elements u and v such that $(u)(v)\subseteq Q, u\notin Q$ and $v\notin\tilde{Q}$. Hence we have $Q\subsetneqq Q:(v)$ and $Q\subsetneqq Q+(v)^n$ for every positive integer n, a contradiction.

§3. Now we shall investigate the previous conditions in the case that the McCoy's radical of every ideal coincides with the nilpotent radical.

Lemma 2. *For an ideal* A *of* R, $\tilde{A}=\tilde{\tilde{A}}$ *if and only if* $\tilde{A}$ *is a semi-prime ideal.*

Proof. If $\tilde{A}=\tilde{\tilde{A}}$ and $\tilde{A}$ is not semi-prime, there exists, by 4.12 Theorem in [1], an element b such that $b\notin\tilde{A}$ and $(b)^2\subseteq\tilde{A}$. Hence we have $b\in\tilde{\tilde{A}}=\tilde{A}$, a contradiction. Conversely, if $\tilde{A}$ is semi-prime and $b\in\tilde{\tilde{A}}$, then there exists a positive integer k such that $(b)^k\subseteq\tilde{A}$. Hence we have $b\in\tilde{A}$ by 4.12 Theorem in [1].

Proposition 1. *For every ideal* A *of* R, $\tilde{A}=\tilde{\tilde{A}}$ *if and only if* $\tilde{A}=\bar{A}$.

Proof. "If part" is immediate. As to "only if part", since $\tilde{A}$ is semi-prime by Lemma 2, $\tilde{A}$ is an intersection of prime ideals. On the other hand, $\tilde{A}\subseteq\bar{A}=\bigcap_{i}\{P_i\mid P_i\text{: a prime ideal containing }A\}$. Therefore we obtain easily $\tilde{A}=\bar{A}$.

Remark. For instance [3], if $(a)(b)$ is finitely generated for any elements a and b of R, we obtain easily $\tilde{A}=\tilde{\tilde{A}}$. Hence, as is well known, in commutative rings or in rings with the maximum condition for ideals, we have $\tilde{A}=\bar{A}$.

In the following, for an ideal B, J_B will mean the set of the indices of the ideals B_α with $\tilde{B}_\alpha=\tilde{B}$.

Lemma 3. *If* N *is a finitely generated ideal, then* $\bigcap_{\alpha\in J_N}(A+N_\alpha)=\bigcap_{n=1}^{\infty}(A+N^n)$.

Proof. It is immediate that $\tilde{N}^n=\tilde{N}$ for any positive integer n.

Hence we have $\bigcap_{\alpha\in J_N}(A+N_\alpha)\subseteq\bigcap_{n=1}^{\infty}(A+N^n)$. Conversely, from $\tilde{N}_\alpha=\tilde{N}$ we have $\tilde{N}_\alpha\supseteq N$. Since N is finitely generated, $N^{k(\alpha)}\subseteq N_\alpha$ for some positive integer $k(\alpha)$. Hence we obtain $\bigcap_{\alpha\in J_N}(A+N_\alpha)\supseteq\bigcap_{n=1}^{\infty}(A+N^n)$.

Theorem 3. *If $\tilde{A}=\tilde{\tilde{A}}$ for every ideal A of R, then the following conditions are equivalent:*

(1) *R is in the class $\mathfrak{D}$ with respect to the McCoy's* (*nilpotent*) *radical.*

(2) (2) *in Theorem* 1.

(3) (3) *in Theorem* 1.

(4) (4) *in Theorem* 1.

(5) (2) *in Theorem* 2.

(6) (3) *in Theorem* 2.

(7) (4) *in Theorem* 2.

(8) (5) *in Theorem* 2.

(9) $A=\bigcap_{n=1}^{\infty}(A+N^n)\cap(\bigcup_{n=1}^{\infty}(A:N^n))$ *for any ideal A and any finitely generated ideal N.*

(10) $A=\bigcap_{n=1}^{\infty}(A+(b)^n)\cap(\bigcup_{n=1}^{\infty}(A:(b)^n)$ *for any ideal A and any element b.*

Proof. By Theorems 1 and 2, it is immediate that conditions (1), $\cdots$, (8) are equivalent. Now we shall prove the theorem by the following implications: (4)$\Rightarrow$(9)$\Rightarrow$(10)$\Rightarrow$(8).

(4)$\Rightarrow$(9): By (4) we have $A=\bigcap_{\alpha\in I_N}(A+N_\alpha)\cap(\bigcup_{\alpha\in I_N}(A:N_\alpha))=\bigcap_{\alpha\in J_N}\cdot(A+N_\alpha)\cap(\bigcup_{\alpha\in J_N}(A:N_\alpha))$. By Lemma 3 we have $A\supseteq\bigcap_{n=1}^{\infty}(A+N^n)\cap(\bigcup_{n=1}^{\infty}(A:N^n))\supseteq A$. Hence we obtain $A=\bigcap_{n=1}^{\infty}(A+N^n)\cap(\bigcup_{n=1}^{\infty}(A:N^n))$.

(9)$\Rightarrow$(10) and (10)$\Rightarrow$(8): These are immediate.

References

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