4. *A Note on Locally Uniform Rings and Modules*

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In [3] and [4], A. W. Goldie has investigated the structure of closed right ideals and annihilator right ideals of (semi)-prime right Goldie rings and has obtained interesting results. We generalize, in Section 1, Goldie's results on closed right ideals and annihilator right ideals of (semi)-prime right Goldie rings to those of right stable rings in the sense of [8]. In second section we shall give "density theorem" in basic uniform modules. Concerning the terminology we refer to [9].

1. On closed right ideals of right stable rings. Let M be a faithful right R-module. A submodule U is said to be uniform iff $U\neq 0$ and every pair of nonzero submodules of U has a nonzero intersection. A submodule K is said to be closed if it has no essential extensions in M. Clearly K is closed iff K is a complemented submodule in the sense of Goldie [4]. An R-module M is said to be locally uniform if every nonzero submodule of M contains a uniform submodule.

Proposition 1. *Let M be a faithful locally uniform right R-module and let K be a closed submodule of M. Then K is an intersection of maximal closed submodules of M* (cf. [4], Theorem 1.5).

Proof. Let K be a relative complement of a submodule L (cf. [4]). Then, there exists an independent set $\{A_i\}$ of uniform submodules such that $L\supset \Sigma_i \oplus A_i$. We set $N_i = K\oplus \Sigma_{j\neq i}\oplus A_j$ for each i, then $N_i\cap A_i=0$. Choose a maximal closed submodule N_i^* such that $N_i^*\supseteq N_i$ and $N_i^*\cap A_i=0$ for each i. If $(\bigcap_i N_i^*)\cap(\Sigma_i\oplus A_i)\neq 0$, then there exist $\{A_i\}_{i=1}^n$ such that $(N_1^*\cap\cdots\cap N_n^*)\cap(A_1\oplus\cdots\oplus A_n)\neq 0$. On the other hand we have $(N_1^*\cap\cdots\cap N_n^*)\cap(A_1\oplus\cdots\oplus A_n)=0$, which is shown by repeated application of the modular law. Hence $(\bigcap_i N_i^*)\cap(\Sigma_i\oplus A_i)=0$ and $K=\bigcap_i N_i^*$, as desired.

Following R. E. Johnson [8], R is said to be a right stable ring iff R is a right locally uniform ring with $Z_r(R)=0$ and $(\Sigma A)^r=0$, where A runs over all uniform right ideals. An element u of R is said to be uniform iff uR^1 is a uniform right ideal, where uR^1 is the principal right ideal generated by u (cf. [4]).

Proposition 2. *If R is a right stable ring, then a right ideal M is a maximal right annihilator ideal if and only if $M=u^r$ for some uniform element u of R. In particular, u^r is maximally closed.*

Proof. The "only if" part is immediate by Theorem 6.9 of [7].

Suppose that M is a maximal annihilator. Then there exists a uniform right ideal A such that $AM^l \neq 0$, because R is a right stable ring. For $0 \neq u \in A \cap M^l$, u^r is maximally closed (by Theorem 6.9 of [7]) and $u^r \supseteqq M$. Hence we have $u^r = M$, as desired.

Proposition 3. *Let R be a right stable ring and let $\hat{R}$ be the maximal right quotient ring of R. If $\hat{R}$ is a left quotient ring of R, then every right annihilator I of R is of the form $\bigcap_i (u_i)^r$, where u_i are uniform elements.*

Proof. By Theorem 2.2 of [10], $L_r^*(R) = J_r^*(R)$. Hence the assertion follows immediately by Propositions 1 and 2.

Proposition 4. *Let R be a finite dimensional right stable ring. Then every proper right annihilator I of R is of the form $u_1^r \cap \cdots \cap u_k^r$, where u_i are uniform elements.*

Proof. Let K be a relative complement of I. Choose a uniform right ideal $A_1 \subseteqq K$. If $I^l A_1 = 0$, then $I \supseteqq A_1$. This is a contradiction. Hence $I^l A_1 \neq 0$. There exists a uniform right ideal C_1 such that $C_1 I^l A_1 \neq 0$, because R is a right stable ring. Hence there exists an element u_1 of $I^l \cap C_1$ such that $u_1 A_1 \neq 0$ and therefore $u_1^r \cap A_1 = 0$, $u_1^r \supseteqq I$. If $u_1^r \cap K = 0$, then clearly $I = u_1^r$. Otherwise we choose a uniform right ideal A_2 in $u_1^r \cap K$. By the same argument as above, there exists a uniform element u_2 of R such that $u_2^r \cap A_2 = 0$ and $u_2^r \supseteqq I$. Since $u_1^r \supseteqq A_2$ and $u_2^r \cap A_2 = 0$, we have $u_1^r \supsetneqq u_1^r \cap u_2^r$. If $u_1^r \cap u_2^r \cap K = 0$, then we obtain $I = u_1^r \cap u_2^r$. Otherwise we choose a uniform right ideal A_3 in $u_1^r \cap u_2^r \cap K$ and a uniform element u_3 of R such that $u_3^r \supseteqq I$ and $u_3^r \cap A_3 = 0$. Clearly $u_1^r \cap u_2^r \supsetneqq u_1^r \cap u_2^r \cap u_3^r$. The process is continued until it terminates, which must occur after not more than $\dim_R R$ terms, because the chain $u_1^r \supsetneqq u_1^r \cap u_2^r \supsetneqq u_1^r \cap u_2^r \cap u_3^r \supsetneqq \cdots$ can not have more than $\dim_R R$ terms. Hence there is an integer $k > 0$ such that $(u_1^r \cap \cdots \cap u_k^r) \cap K = 0$ and $(u_1^r \cap \cdots \cap u_k^r) \supseteqq I$. Hence we obtain $I = u_1^r \cap u_2^r \cap \cdots \cap u_k^r$.

2. Density Theorem in basic uniform R-modules. Throughout this section, the ring R will be a right and left locally uniform prime ring with $Z_r(R) = Z_l(R) = 0$. Let M be a right R-module. The set $Z_R(M) = \{m \in M \mid m^r \subset' R\}$ is a submodule called the singular submodule of M, where $m^r = \{a \in R \mid ma = 0\}$. As in [5], an R-module M is said to be *basic* if

(i) $Z_R(M) = 0$, and

(ii) for each nonzero submodule N, there exists an R-monomorphism ϕ such that $\phi : M \to N$.

If M is a locally uniform basic R-module, then M is uniform and M is a prime module in the sense of [2].

The followings are examples of uniform basic R-modules.

(i) If R is a right and left locally uniform prime ring with $Z_r(R)$

$=Z_1(R)=0$ and if M is a uniform right ideal of R, then M is a basic uniform prime R-module.

(ii) Let R be a semi-prime ring and let M be a torsion-less R-module in the sense of [11]. If M is a uniform R-module with $Z_R(M)=0$, then M is a basic uniform R-module. In particular, if R is a prime ring, then M is a prime R-module.

Now, let M be a basic uniform R-module and let $K=\mathrm{Hom}_R(M, M)$. By Lemma 5.4 of [5], nonzero elements of K are non-singular mappings and hence M is a torsion-free left K-module. Since $Z_R(M)=0$, there exists a uniform right ideal U of R such that $MU\neq 0$. Hence $mU\neq 0$ for some $m\in M$. Then, by Theorem 2.4 of [2], we obtain $mU\cong U$. Let $\hat{R}$ be the maximal right quotient ring of R. We set $\hat{U}=E_R(U)$ in $\hat{R}$, where $E_R(U)$ is an injective hull of U. Then $\hat{U}$ is a minimal right ideal of $\hat{R}$ and $\Delta=\mathrm{Hom}_{\hat{R}}(\hat{U}, \hat{U})$ is the right quotient division ring of $\Gamma=\mathrm{Hom}_R(U, U)$ by Theorem 1.2 of [1; p. 97]. Since $E_R(M)=E_R(mU)\cong E_R(U)=\hat{U}$, we may assume that M is an R-submodule of $\hat{U}$. Since M is basic, there exists an R-monomorphism σ such that

$$\sigma: M \longrightarrow U.$$

Clearly, there exists a uniform left ideal W such that $WU\neq 0$ and $UW\neq 0$. Then $D=W\cap U$ is a left and right Ore domain and we obtain the natural inclusions $D\subset\Gamma\subset K$ as abelian groups, where $D\to\Gamma$ is a left multiplication and $\phi: \Gamma\to K$ is defined by $\phi(\alpha)=\alpha\sigma$ for $\alpha\in\Gamma$.

Lemma 1. *If x and y are nonzero elements of M, then $Kx\cap Ky\neq 0$ if and only if $x^r=y^r$.*

Proof. Since $Z_R(M)=0$ and M is uniform, x^r is a maximal closed right ideal of R for every nonzero element x of M, by Theorem 6.9 of [7]. Hence the "if" part is clear. Conversely suppose that $x^r=y^r$. We set $x'=\sigma(x)$ and $y'=\sigma(y)$. Then $(x')^r=(y')^r$ and $(x')^{rl}=(y')^{rl}$ is a minimal annihilator left ideal of R. Hence $Wx'\cap Wy'\neq 0$ and $D(Wx'\cap Wy')\neq 0$, because R is a prime ring. There exist elements $d\in D$; $b, b_1\in W$ such that $0\neq dbx'=db_1y'$. Then clearly $db, db_1\in D$ and therefore $0\neq(db\sigma)x=dbx'=db_1y'=(db_1\sigma)y$ for $db\sigma, db_1\sigma\in K$.

As usual, the elements $x_1, \cdots, x_n$ of M are called K-linearly independent if and only if $k_1x_1+\cdots+k_nx_n=0$ implies that all $k_i=0$, $k_i\in K$.

The following lemma follows from the same arguments as in Lemma 3.1 of [6].

Lemma 2. *The elements $x_1, \cdots, x_n$ of M are K-linearly independent if and only if $(x_j)^r\not\supseteq\bigcap_{i=1, i\neq j}^n (x_i)^r$, $j=1, \cdots, n$.*

Theorem 2 (Density theorem in basic uniform modules). *If $[x_1, \cdots, x_n]$ is any set of K-linearly independent elements of M and if $[y_1, \cdots, y_n]$ is any set of n elements of M, then there exists an element a of*

R and a nonzero element k of K such that

$$x_i a = k y_i, \qquad i=1, \cdots, n.$$

Proof. Let q be an element of $\hat{R}$ and let $L_q=\{r \in R \mid qr \in U\}$. Then for each x_i, $L_{x_i} \subset' R$ as a right R-module, because $M' \supset U$ as a right R-module. We now set $I_i = \bigcap_{j=1, i \neq j}^{n} (x_j)^r$ and $I_i' = I_i \cap L_{x_i}$. Then $I_i' \supset I_i'$ as a right R-module. Hence we obtain $x_i I_i' \neq 0$ and $x_i I_j' = 0$ $(i \neq j)$ by Lemma 2. Since $x_i I_i'$ is a right ideal of R, we have $(x_i I_i')^r = 0$. Hence there exist elements $e_i \in D$ and $b_i \in I_i'$ such that $x_i b_i e_i y_i \neq 0$ for all $y_i' \neq 0$, where $y_i' = \sigma(y_i)$. We set $a_i = b_i e_i y_i'$ for all $y_i' \neq 0$ and $a_j = 0$ for all $y_j' = 0$. Then $x_i a_i = x_i b_i e_i y_i' = d_i y_i'$, where $d_i = x_i b_i e_i \in D$. Now, for all $y_i' \neq 0$, $d_i y_i' W \neq 0$ and hence $d_i y_i' W \neq 0$ for some $w_i \in W$, $d_i y_i' w_i \in UW \subseteq U \cap W = D$. Since D is a right Ore domain, $d_i y_i' w_i D$ is a nonzero right ideal of D for each $y_i' \neq 0$ and $\bigcap_i d_i y_i w_i D \neq 0$. Select an element d such that $d \in d_i y_i' w_i D$, $d \neq 0$ and $d = d_i y_i' w_i c_i$ for each $y_i' \neq 0$. Then, putting $a = a_1 w_1 c_1 y_1' + \cdots + a_n w_n c_n y_n'$, we obtain $x_i a = d y_i' = (d\sigma) y_i$, for $d\sigma \in K$, as desired.

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