3. A Note on Artinian Subrings

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Throughout, A will represent a ring with the identity element 1, $J(A)$ the radical of A, and a subring of A will mean one containing 1. If S is a subset of A, $V_A(S)$ means the centralizer of S in A. A left A-module M is always unital and denoted by ${}_AM$.

The purpose of this note is to prove the following:

Theorem 1. *Let B be a subring of A such that ${}_BA$ is f.g. (finitely generated), and T a left Artinian subring of A containing B. Let $\bar{T}=T/J(T)$, and $\bar{B}=B+J(T)/J(T)$. If $\bar{T}=\bar{B}\cdot V_{\bar{T}}(\bar{B})$ and the left $\bar{T}$-module $\bar{A}=A/J(T)A$ is faithful then B is left Artinian.*

Our theorem contains evidently D. Eisenbud [3; Theorem 1 b)] and draws out J.-E. Björk [2; Theorem 3.4] as an easy corollary.

Lemma 1. *Let $M=Au_1+Au_2+\cdots+Au_n$ be a unital A-A-module such that $Au_i=u_iA$ and u_1 is left A-free. If for every non-zero ideal $\mathfrak{a}$ of A there holds $\mathfrak{a}M=M$, then A is two-sided simple.*

Proof. Without loss of generality, we may assume that $M\neq Au_1+\cdots+Au_{i-1}+Au_{i+1}+\cdots+Au_n$ for each $1<i\leqslant n$. We shall prove then by induction $M=Au_1\oplus\cdots\oplus Au_n$, which implies at once that A is two-sided simple. We set $M_k=Au_1+\cdots+Au_k$ for $1\leqslant k\leqslant n$. Evidently, $\mathfrak{a}_n=\{a\in A\mid au_n\in M_{n-1}\}$ is an ideal of A. If $\mathfrak{a}_n$ is non-zero then $M=\mathfrak{a}_nM=M_{n-1}$. This contradiction proves $M=M_{n-1}\oplus Au_n$. Next, assume that $M=M_k\oplus Au_{k+1}\oplus\cdots\oplus Au_n$ has been proved. It will be easy to see that $M_k\neq Au_1+\cdots+Au_{i-1}+Au_{i+1}+\cdots+Au_k$ for each $1<i\leqslant k$. If $\mathfrak{a}$ is a non-zero ideal of A then $\mathfrak{a}M_k\oplus\mathfrak{a}u_{k+1}\oplus\cdots\oplus\mathfrak{a}u_n=M_k\oplus Au_{k+1}\oplus\cdots\oplus Au_n$ implies at once $\mathfrak{a}M_k=M_k$. Hence, by the first step, we obtain $M_k=M_{k-1}\oplus Au_k$, which completes the induction.

Proposition 1. *Let $A=A_1\oplus\cdots\oplus A_n$, where A_i is a two-sided simple [Artinian simple] ring with the identity element e_i. Let B be a subring of A such that ${}_BA$ is f.g. If $A=B\cdot V_A(B)$ then $V_A(B)=V_1\oplus\cdots\oplus V_n$ and $B=B_1\oplus\cdots\oplus B_k$ $(k\leqslant n)$, where V_i is Artinian simple and B_i is two-sided simple [Artinian simple].*

Proof. At first, we shall prove the case $A=A_1$. Evidently, $A=Bv_1+\cdots+Bv_s$ with $v_1=1$ and $v_2,\cdots,v_s\in V_A(B)$. As we can easily

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see that $\mathfrak{b}A=A$ for every non-zero ideal $\mathfrak{b}$ of B, Lemma 1 proves that B is two-sided simple. Accordingly, $A=B\cdot V_A(B)=B\otimes_Z V_A(B)$, where Z is the center of B. Recalling that A is two-sided simple [Artinian simple] and f.g. over B, it is obvious that $V_A(B)$ is a simple algebra of finite rank over Z [and B is Artinian simple].

Next, we shall prove the general case. Obviously, B is a subdirect sum of $B'_i=e_iB$ $(i=1,\cdots,n)$ and ${}_{B'_i}A_i$ is f.g. Since $A=B\cdot V_A(B)$ implies $A_i=B'_i\cdot V_{A_i}(B'_i)$, the case $n=1$ proves that $V_{A_i}(B'_i)$ is Artinian simple and B'_i is two-sided simple [Artinian simple]. Now, the rest of the proof will be evident.

Lemma 2. *Let B be a subring of A. If there exists a left free A-module M such that ${}_BM$ is f.g., then ${}_BA$ is f.g.*

Proof. Let $M=\sum_{i=1}^m Bu_i$, and $\{w_1,\cdots,w_n\}$ a free basis of ${}_AM$. Since $u_i=\sum_{j=1}^n a_{ij}w_j$ with some $a_{ij}\in A$, we obtain $(\sum_{i,j}Ba_{ij})w_1\oplus\cdots\oplus(\sum_{i,j}Ba_{ij})w_n=M$, whence it follows $A=\sum_{i,j}Ba_{ij}$.

The next is [2; Theorem 1.3], whose proof given in [2] is rather long and due to a lemma of Osofsky. We shall present here a notably short, elementary proof.

Lemma 3. *Let A be left Artinian, and B a subring of A. If $\bar{B}=B+J(A)/J(A)$ is left Artinian and $\bar{A}=A/J(A)$ is left f.g. over $\bar{B}$, then B is left Artinian.*

Proof. Since ${}_{\bar{B}}\bar{B}$ satisfies both chain conditions and $J(A)^h/J(A)^{h+1}$ is left f.g. over $\bar{B}$, $J(A)^h/J(A)^{h+1}$ has a composition series as left B-module. Accordingly, $J(A)$ being nilpotent, ${}_BA$ satisfies both chain conditions, whence it follows that B is left Artinian.

Now, we are at the position to prove Theorem 1.

Proof of Theorem 1. Left $\bar{T}=T_1\oplus\cdots\oplus T_n$, where T_i is an Artinian simple ring with the identity element e_i. Then, $\bar{B}$ is a subdirect sum of $B'_i=e_i\bar{B}$ $(i=1,\cdots,n)$ and $\bar{A}=e_1\bar{A}\oplus\cdots\oplus e_n\bar{A}$ is the decomposition of (completely reducible) ${}_{\bar{T}}\bar{A}$ into the homogeneous components. Since ${}_{\bar{T}}\bar{A}$ is faithful, each $e_i\bar{A}$ is an f.g. non-zero left B'_i-module and there exists a positive integer n_i such that $M_i=(e_i\bar{A})^{(n_i)}$ (the direct sum of n_i copies of $e_i\bar{A}$) is left T_i-free. Then, Lemma 2 proves that ${}_{B'_i}T_i$ is f.g., whence it follows that ${}_{\bar{B}}\bar{T}$ is f.g. Since $\bar{T}=\bar{B}\cdot V_{\bar{T}}(\bar{B})$, Proposition 1 proves then $\bar{B}$ is Artinian. It follows therefore by Lemma 3 that B is left Artinian.

Corollary 1 ([3; Theorem 1b)]). *Let A be left Artinian, and B a subring of A such that ${}_BA$ is f.g. If $A=B\cdot V_A(B)$ then B is left Artinian.*

Corollary 2 ([2; Theorem 3.4]). *Let A be left Artinian, and B a subring of A such that ${}_BA$ is f.g. If $A/J(A)$ is a direct sum of division rings and f.g. over its center C, then B is left Artinian.*

Proof. Let $\bar{A}=A/J(A)$, and $\bar{B}=B+J(A)/J(A)$. Since C is Artinian and ${}_C\bar{A}$ is f.g., $C\cdot\bar{B}$ is Artinian, and hence a direct sum of division rings. If T is the inverse image of $C\cdot\bar{B}$ with respect to the natural homomorphism of A onto $\bar{A}$, then $J(T)$ coincides with $J(A)$ and T is left Artinian by Lemma 3. Hence, B is left Artinian by Theorem 1.

Corollary 3. *Let A be a division ring, and B a subring of A such that ${}_BA$ is f.g. If B satisfies a polynomial identity then B is a division ring. In particular, if B is commutative then B is a field.*

Proof. By [1; Theorem 1], B is a (right and left) Ore domain and the quotient division ring of B contained in A is finite over its center. Then, to be easily seen, A satisfies a standard identity. Accordingly, A is finite over its center again by [1; Theorem 1], and then B is a division ring by Corollary 2.

References

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