43. On Closed Mappings of Generalized Metric Spaces

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N. S. Lašnev [5] proved the following theorem:

Let f be a closed continuous map (=mapping) from a metric space X onto a space Y. Then $Y=\bigcup_{n=0}^{\infty} Y_n$, *where* Y_n *is a discrete subset of Y for each* $n>1$, *and* $f^{-1}(y)$ *is compact for each* $y \in Y_0$.

This theorem was extended by several mathematicians, especially by A. Okuyama [7] to normal σ-space, by R. A. Stoltenberg [9] to normal semi-stratifiable space and by V. V. Filippov [1] to paracompact M-space. It is almost a surprising fact that under such general circumstances so many points of Y have compact inverse images. Besides, this type of theorem is not a mere object of curiosity as shown by F. G. Slaughter [8] who used it to prove an interesting metrization theorem. Since, as well known, a regular space is metrizable if and only if it is σ and M and since every σ-space is semi-stratifiable, the above theorems by Okuyama and Filippov are extensions of Lašnev's theorem to two different directions while Stoltenberg's theorem generalizes Okuyama's. Thus it is natural to try to unify the two theorems of Stoltenberg and of Filippov. Although this attempt is not fully successful yet, we have made a partial success. In fact the purpose of this paper is to extend Lašnev's theorem to two classes of generalized metric spaces which contain all M-spaces as well as all semi-metric spaces. (Note that semi-metric = semi-stratifiable plus 1-st countable as proved by R. W. Heath [2].)

All spaces in the following discussions are T_1 except in Definitions, and all maps are continuous. N denotes the sequence of natural numbers $\{1,2,3,\cdots\}$, and a subsequence means an infinite subsequence. As for general terminologies and symbols, see J. Nagata [6] and also the above mentioned references for more specialized terminologies.

Let us recall that a collection $\mathcal{U}$ of (not necessarily open) subsets of a space X is called a *nbd base of a subset F* of X if the interior of each member of $\mathcal{U}$ contains F and if for every open set V containing F there is a member of $\mathcal{U}$ which is contained in V and also that a collection $\{F_\alpha \mid \alpha \in A\}$ of subsets of X is called to be *cushioned* in a collection $\{U_\alpha \mid \alpha \in A\}$ of subsets if $(\cup\{F_\alpha \mid \alpha \in A'\})^- \subset \cup\{U_\alpha \mid \alpha \in A'\}$ for every subset A' of A. One of our new classes of generalized metric spaces is defined as follows.

Definition 1. A topological space X is called a *QSM-space* if X has a cover $\{F_\alpha \mid \alpha \in A\}$ by countably compact closed subsets and if each F_α has a nbd base $\{U_{\alpha n} \mid n=1,2,\cdots\}$ such that for each n $\{F_\alpha \mid \alpha \in A\}$ is cushioned in $\{U_{\alpha n} \mid \alpha \in A\}$. (Each F_α is called a *kernel*.)

Proposition 1. *All semi-metric spaces as well as all M-spaces are QSM.*

Proof. If X is semi-metric, then let $F_x=\{x\}$ for each $x \in X$, and $U_{xn}=$the spherical nbd of x with radius $1/n$. If X is an M-space, then it has a sequence $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \cdots$ of open covers such that every point sequence $\{x_i \mid i=1,2,\cdots\}$ satisfying $x_i \in S(x_0, \mathcal{U}_i)$, $i=1,2,\cdots$ has a cluster point. Thus $F_x=\bigcap_{n=1}^\infty S(x,\mathcal{U}_n)$ for each $x \in X$ and $U_{xn}=S(x,\mathcal{U}_n)$ satisfy the conditions in Definition 1.

In the above two cases it is easy to see that if x is a cluster point of a sequence $\{F_{x_1}, F_{x_2}, \cdots\}$ of members of $\{F_x \mid x \in X\}$, then it is a cluster point of $\{x_i\}$. Thus the following theorem implies Stoltenberg's theorem in the special case that X is semi-metric and also Filippov's theorem.

Theorem 1. *Let X be normal and QSM and f a closed map from X onto a space Y. Then $Y=\bigcup_{n=0}^\infty Y_n$, where each Y_n is discrete for $n>1$, and for each $y \in Y_0$ $f^{-1}(y)$ is k-countably compact, i.e. every sequence $\{F_{\alpha(i)} \mid i=1,2,\cdots\}$ of kernels has a cluster point if $F_{\alpha(i)} \cap f^{-1}(y) \neq \emptyset$, $i=1,2,\cdots$.*

Proof. Let $\{U_{\alpha n}\}$ and $\{F_\alpha\}$ satisfy the conditions in Definition 1. We may assume that $U_{\alpha n} \supset U_{\alpha n+1}$ for each α and n. We shall call a point sequence $\{y_i \mid i=1,2,\cdots\}$ *distinct* if $y_i \neq y_j$ for $i \neq j$. Put $Y_n=\{y \in Y \mid$ For every distinct point sequence $\{y_i \mid i=1,2,\cdots\}$ in Y, there is $\alpha \in A$ such that $F_\alpha \cap f^{-1}(y) \neq \emptyset$ and $U_{\alpha n} \cap (\cup\{f^{-1}(y_i) \mid i \in I\})^- = \emptyset$ for some (infinite) subsequence I of $N\}$, $n=1,2,\cdots$, and

$$Y_0=Y-\bigcup_{n=1}^\infty Y_n.$$

Let us prove that Y_n is discrete for each $n>1$. Assume the contrary; then the sequence $\{f^{-1}(y) \mid y \in Y_n\}$ has a cluster point $x \in X$, because the map f is closed. Let $x \in F_\alpha$; then there is a distinct point sequence $\{y_i \mid i=1,2,\cdots\}$ such that $f^{-1}(y_i) \cap U_{\alpha i} \neq \emptyset$, $i=1,2,\cdots$. Then $\{f^{-1}(y_i) \mid i=1,2,\cdots\}$ has no discrete subsequence as seen in the following. Let $\{i(k) \mid k=1,2,\cdots\}$ be any subsequence of N. If $f^{-1}(y_{i(k)}) \cap F_\alpha \neq \emptyset$ for infinitely many k, then $\{f^{-1}(y_{i(k)}) \mid k=1,2,\cdots\}$ clusters because of the countable compactness of F_α. If $f^{-1}(y_{i(k)}) \cap F_\alpha = \emptyset$ for $k \geq k_0$, then the set $F=\cup\{f^{-1}(y_{i(k)}) \mid k \geq k_0\}$ is disjoint from F_α. Since $F \cap U_{\alpha k} \neq \emptyset$ for all k, F cannot be closed. Therefore $\{f^{-1}(y_{i(k)}) \mid k=1,2,\cdots\}$ is not discrete. But from another point of view, as seen in the following, we can argue that it is possible to select a discrete subsequence from $\{f^{-1}(y_i)\}$. Since $y_1 \in Y_n$, there is $\alpha(1) \in A$ such that $F_{\alpha(1)} \cap f^{-1}(y_1) \neq \emptyset$ and a subsequence I_2 of $I_1=N$ such that $U_{\alpha(1)n}\cap(\cup\{f^{-1}(y_i)\mid i\in I_2\})^-=\emptyset$. Choose $i(2)\in I_2$ satisfying $i(2)>i(1)=1$. Since $y_{i(2)}\in Y_n$, there is $\alpha(2)\in A$ such that $F_{\alpha(2)}\cap f^{-1}(y_{i(2)})\neq\emptyset$ and a subsequence I_3 of I_2 such that $U_{\alpha(2)n}\cap(\cup\{f^{-1}(y_i)\mid i\in I_3\})^-=\emptyset$. Choose $i(3)\in I_3$ satisfying $i(3)>i(2)$. Repeating this process we obtain a subsequence $i(1)<i(2)<i(3)<\cdots$ of N such that

$$F_{\alpha(k)}\cap f^{-1}(y_{i(k)})\neq\emptyset,$$

$$(1)\qquad U_{\alpha(k)n}\cap(\cup\{f^{-1}(y_{i(j)})\mid j>k\})^-=\emptyset,\qquad k=1,2,\cdots$$

Select $x_k\in F_{\alpha(k)}\cap f^{-1}(y_{i(k)})$, $k=1,2,\cdots$. Then the distinct sequence $\{x_k\}$ does not cluster. Because if x is a cluster point of $\{x_k\}$, then $x\in(\bigcup_{k=1}^{\infty}F_{\alpha(k)})^-$, and also by (1) $x\notin\bigcup_{k=1}^{\infty}U_{\alpha(k)n}$. But this contradicts the fact that $\{F_\alpha\}$ is cushioned in $\{U_{\alpha n}\}$. Thus $\{x_k\}$ is a discrete sequence, and hence $\{f^{-1}(y_{i(k)})\mid k=1,2,\cdots\}$ is also discrete contradicting the previous argument. Therefore Y_n must be discrete.

Now, let us prove that $f^{-1}(y)$ is k-countably compact for each $y\in Y_0$. Let $\{F_{\alpha(i)}\mid i=1,2,\cdots\}$ be a sequence of kernels such that $F_{\alpha(i)}\cap f^{-1}(y)\neq\emptyset$, $i=1,2,\cdots$. Assume that $\{F_{\alpha(i)}\mid i=1,2,\cdots\}$ does not cluster; then since each kernel is countably compact and closed and X is normal, there is a subsequence of $\{\alpha(i)\mid i=1,2,\cdots\}$, which we denote again by $\{\alpha(i)\}$, and a discrete sequence $\{U_i\mid i=1,2,\cdots\}$ of open sets such that $F_{\alpha(i)}\subset U_i$. Then there is a subsequence $n(1)<n(2)<\cdots$ of N such that $U_{\alpha(i)n(i)}\subset U_i$, $i=1,2,\cdots$. For brevity of symbols we denote the discrete sequence $\{U_{\alpha(i)n(i)}\mid i=1,2,\cdots\}$ by $\{U_{\alpha(i)i}\mid i=1,2,\cdots\}$. Now, for each $n\in N$ choose $i(n)\in N$ such that $i(1)<i(2)<\cdots$ and such that

$$(2)\qquad U_{\alpha(n)i(n)}\subset U^0_{\alpha(n)n},\qquad n=1,2,\cdots,$$

$$(3)\qquad U_{\alpha(1)i(n)}\subset U^0_{\alpha(1)n},\qquad n=1,2,\cdots,$$

where U^0 denotes the interior of U.

On the other hand, since $y\notin Y_n$ for each $n>1$, there is a distinct sequence $\{y_i^n\mid i=1,2,\cdots\}$ such that for every α satisfying $F_\alpha\cap f^{-1}(y)\neq\emptyset$ and for every subsequence I of N,

$$(4)\qquad U_{\alpha n}\cap(\cup\{f^{-1}(y_i^n)\mid i\in I\})^-\neq\emptyset.$$

We claim that for each $n\in N$ there is $j(n)$ such that

$$(5)\qquad U_{\alpha(n)n}\cap f^{-1}(y_j^{i(n)})\neq\emptyset\qquad\text{for all } j\geq j(n),$$

$$(6)\qquad U_{\alpha(1)n}\cap f^{-1}(y_j^{i(n)})\neq\emptyset\qquad\text{for all } j\geq j(n).$$

Because, e.g. if we deny (5), then $U_{\alpha(n)n}\cap(\cup\{f^{-1}(y_j^{i(n)})\mid j\in I\})=\emptyset$ for some subsequence I of N, which combined with (2) implies $U_{\alpha(n)i(n)}\cap(\cup\{f^{-1}(y_j^{i(n)})\mid j\in I\})^-=\emptyset$ contradicting (4). Similarly (6) follows from (3) and (4). Observe that we can choose $j(n)$ in such a way that $j(1)<j(2)<\cdots$, and $\{y_{j(n)}^{i(n)}\mid n=1,2,\cdots\}$ is a distinct sequence. Select $z_n\in U_{\alpha(n)n}\cap f^{-1}(y_{j(n)}^{i(n)})$, $n=1,2,\cdots$ (Recall (5)). Then the discreteness of $\{z_n\}$ follows from that of $\{U_{\alpha(n)n}\mid n=1,2,\cdots\}$. Hence $\{f^{-1}(y_{j(n)}^{i(n)})\mid n=1,2,\cdots\}$ is also discrete, because f is a closed map. Thua st most finitely many of $f^{-1}(y^{i(n)}_{j(n)})$ intersect $F_{\alpha(1)}$, and hence there is $k \in N$ for which $U_{\alpha(1)k} \cap f^{-1}(y^{i(k)}_{j(k)}) = \emptyset$. But this contradicts (6), and thus $f^{-1}(y)$ is k-countably compact.

T. Ishii [3], [4] defined wM-space and extended Filippov's theorem to wM-spaces (F. Slaughter has pointed out that there is a gap in Ishii's proof but the theorem remains true.), where X is a *wM-space* if it has a sequence $\{\mathcal{U}_n \mid n=1,2,\cdots\}$ of open covers such that every point sequence $\{x_n \mid n=1,2,\cdots\}$ satisfying $x_n \in S^2(x_n, \mathcal{U}_n)$, $n=1,2,\cdots$ clusters. We can modify our Theorem 1 a little to let it include Ishii's theorem, too.

Definition 2. A topological space X is called an *SSM space* if each point x of X has a sequence $\{U_{xn} \mid n=1,2,\cdots\}$ of (not necessarily open) nbds satisfying

i) $y \in U_{xn}$ implies $x \in U_{yn}$,

ii) if $\{x_i \mid i=1,2,\cdots\}$ is a discrete point sequence in X, then there are subsequences $\{n(j) \mid j=1,2,\cdots\}$ and $\{i(j) \mid j=1,2,\cdots\}$ of N such that $i(1)=1$, and $\{U_{x_{i(j)}n(j)} \mid j=1,2,\cdots\}$ is discrete.

Proposition 2. *All normal semi-metric spaces as well as all wM-spaces are SSM. Accordingly all spaces listed in the following are SSM: Metric, $M, M^*, M^\sharp$, Nagata (=1-st countable and stratifiable), 1-st countable normal and σ.*

The easy proof is left to the reader.

The proof of the following theorem is quite analogous to that of Theorem 1, so that it will be left to the reader.

Theorem 2. *Let X be SSM and a closed map from X onto a space Y. Then $Y=\bigcup_{n=0}^{\infty} Y_n$, where each Y_n is discrete for $n>1$, and for each $y \in Y_0$ $f^{-1}(y)$ is countably compact.*

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