40. On the Commutation Relation AB−BA=C

By Hiroyuki SUZUKI
Department of Mathematics, Tokyo Metropolitan University

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We shall deal with commutation relation of the infinitesimal generators of strongly continuous semi-groups on a Banach space X.

A few general references for this work are Foias, C., L. Geher and B. Sz.-Nagy [1] and T. Kato [2]. The purpose of this paper is to obtain a generalization of T. Kato's theorem [2]. The proof of the theorem is similar to that of T. Kato's theorem.

The main theorem is as follows.

Theorem. *Let $\{e^{sA}\}$ and $\{e^{tB}\}$ be two contraction semi-groups on a Banach space X satisfying the relation*

$$e^{sA}e^{tB}=e^{tB}e^{tsC}e^{sA} \qquad 0\leqq s,t<\infty \tag{1}$$

for some contraction semi-group $\{e^{uC}\}$ and suppose that $D(C)\supset D(B)$. Then

(a) $\Omega=D(AB)\cap D(BA)$ *is dense in* X

(b) $(AB-BA)x=Cx$ *for* $x\in\Omega$

(c) $(A-a)(B-b)\Omega=X$ *for all* a,b *satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$.

Conversely, let C be the infinitesimal generator of a contraction semi-group. We suppose that $D(C)\supset D(A)$, $D(C)\supset D(B)$ and C commutes with $R(a;A)$ and $R(b;B)$ for some pair a,b satisfying $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, *and that there exists a dense linear subset Ω of $D(AB)\cap D(BA)$ for which* (b) *holds. Furthermore, if we suppose, for some pair a,b satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, *$(A-a)(B-b)\Omega$ is dense in X. Then* (1) *holds.*

Remark. *If the condition $D(C)\supset D(B)$ of the first part of the theorem is replaced by $D(C)\supset D(A)$, then we have, in* (c), *$(B-b)(A-a)\Omega=X$ for all a,b satisfying* $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$.

Proof of the first part. Multiplication of (1) by e^{-bt} followed by an integration with respect to t on $(0,\infty)$ yields

$$e^{sA}(B-b)^{-1}=(B+sC-b)^{-1}e^{sA} \qquad s\geqq 0, \tag{2}$$

whenever $\mathrm{Re}(b)>0$.

Since, for sufficiently small $s>0$, $B+sC$ generates a contraction semi-group by Hille-Yosida's theorem. Differentiation of (2) with respect to s followed by setting $s=0$ leads to

$$A(B-b)^{-1}\supset(B-b)^{-1}A-(B-b)^{-1}C(B-b)^{-1}$$

and hence, for $\mathrm{Re}(a)>0$ and $\mathrm{Re}(b)>0$,

$$(3)\qquad (B-b)^{-1}(A-a)^{-1}=(A-a)^{-1}(B-b)^{-1} -(A-a)^{-1}(B-b)^{-1}C(B-b)^{-1}(A-a)^{-1}.$$

If $y \in X$ and

$$(4)\qquad x=(B-b)^{-1}(A-a)^{-1}y,$$

then

$$y=(A-a)(B-b)x$$

and hence, by (3)

$$x=(A-a)^{-1}(B-b)^{-1}(y-Cx).$$

Hence

$$x \in D((B-b)(A-a)) \quad \text{and} \quad (B-b)(A-a)x=(A-a)(B-b)x-Cx.$$

Thus we have $x \in D(AB)\cap D(BA)\equiv \Omega$ and $(AB-BA)x=Cx$.

It is clear that any element x of Ω can be expressed in the form (3) by letting $y=(A-a)(B-b)x$, and thus relation (b) holds.

Also, since $y \in X$ is arbitrary, then $(A-a)(B-b)\Omega=X$ and $\Omega=(B-b)^{-1}(A-a)^{-1}X$. Since A and B are densely defined, Ω is dense in X.

Proof of the second part. Let a_0, b_0 denote constants for which $\mathrm{Re}(a_0)>0, \mathrm{Re}(b_0)>0$ and $(A-a_0)(B-b_0)\Omega$ is dense in X. If $x \in \Omega$ and $y=(A-a_0)(B-b_0)x$, then by (b)

$$y=(B-b_0)(A-a_0)x+Cx$$

and consequently

$$(B-b_0)^{-1}(A-a_0)^{-1}y=x=(A-a_0)^{-1}(B-b_0)^{-1}(y-Cx)$$
$$=(A-a_0)^{-1}(B-b_0)^{-1}y-(A-a_0)^{-1}(B-b_0)^{-1}C(B-b_0)^{-1}(A-a_0)^{-1}y.$$

Since the y's are dense, (3) holds when $a=a_0$ and $b=b_0$.

Next, it will be shown that

$$(5)\qquad (B-b)^{-n}(A-a)^{-1}=(A-a)^{-1}(B-b)^{-n} -n(A-a)^{-1}(B-b)^{-n}C(B-b)^{-1}(A-a)^{-1}$$

holds for $a=a_0$ and $b=b_0$ and $n=1,2,\cdots$. The assertion has already been established for $n=1$. Now assume that (5) holds for $a=a_0$ and $b=b_0$ and some n.

We put

$$M=(A-a_0)^{-1} \quad \text{and} \quad N=(B-b_0)^{-1}$$

then

$$\begin{aligned} MN^{n+1}-N^{n+1}M &= (MN-NM)N^n+N(MN^n-N^nM) \\ &= MNCNMN^n+nNMN^nCNM \\ &= MNCN(N^nM+nMN^nCNM) \\ &\quad +n(MN-MNCNM)N^nCNM \\ &= (n+1)MN^{n+1}CNM. \end{aligned}$$

In the last equality we use the fact that C commutes with $R(b\,;B)$ for all b, satisfying $\mathrm{Re}(b)>0$. Thus (5) holds for $a=a_0$ and $b=b_0$ and $n=1,2,\cdots$.

Since

$$(B-b)^{-1}=\sum_{k=1}^{\infty}(b-b_0)^{k-1}(B-b_0)^{-k}$$

and

$$(B-b)^{-2}=\sum_{k=1}^{\infty}k(b-b_0)^{k-1}(B-b_0)^{-k-1}$$

it follows from (5) for $a=a_0$ and $b=b_0$ that (3) holds for $a=a_0$ and $|b-b_0|$ sufficiently small. Since $(B-b)^{-1}$ is analytic for $\mathrm{Re}(b)>0$, then (3) must hold for $a=a_0$ and $\mathrm{Re}(b)>0$.

An $(n-1)$-fold differentiation of (when $a=a_0$) (3) with respect to b then shows that (5) holds for $a=a_0$ and $\mathrm{Re}(b)>0$. If (5) is multiplied by $(-b)^n$ and if $b=n/t$ $(t>0)$ then

$$\begin{aligned}(1-n^{-1}tB)^{-n}(A-a)^{-1}&=(A-a)^{-1}(1-n^{-1}tB)^{-n}\\&+t(A-a)^{-1}(1-n^{-1}tB)^{-n}C(1-n^{-1}tB)^{-1}(A-a)^{-1}\end{aligned}$$

for $a=a_0$ and $n>0$. But

$$s-\lim_{n\to\infty}(1-n^{-1}tB)^{-n}=e^{tB}$$

(Hille and Phillips [3], p. 362) and so

$$\begin{aligned}e^{tB}(A-a)^{-1}&=(A-a)^{-1}e^{tB}+t(A-a)^{-1}e^{tB}C(A-a)^{-1}\\&=(A-a)^{-1}e^{tB}(A+tC-a)(A-a)^{-1}\quad\text{for}\quad a=a_0,\quad t\geqq 0.\end{aligned}$$

Here we remark that, since C commutes with $R(a\,;A)$ for some a satisfying $\mathrm{Re}(a)>0$ and $D(C)\supset D(A)$, $\{e^{uC}\}$ commutes with $\{e^{sA}\}$.

Thus the closure of $A+tC$ generates a contraction semi-group $\{T_s=e^{sA}e^{stC}\}$ for all $t>0$ by Trotter's theorem [4]. It follows that

(6) $$e^{tB}(\overline{A+tC}-a)^{-1}=(A-a)^{-1}e^{tB}$$

for $a=a_0$, and hence

(7) $$e^{tB}(\overline{A+tC}-a)^{-n}=(A-a)^{-n}e^{tB}\qquad(n=1,2,\cdots)$$

for $a=a_0$, where $\overline{A+tC}$ is the closure of $A+tC$.

Using the power series representation for $(A-a)^{-1}$ and $(\overline{A+tC}-a)^{-1}$ near $a=a_0$ one concludes from (7) (where $a=a_0$) that (6) holds for $|a-a_0|$ sufficiently small and, by analytic continuation, for all a satisfying $\mathrm{Re}(a)>0$. A differentiation of (6) with respect to a shows that (7) holds also for $\mathrm{Re}(a)>0$.

If one multiplies both sides of (7) by $(-a)^n$, let $a=n/s$ where $s>0$, and then let $n\to\infty$, one obtains $e^{sA}e^{tB}=e^{tB}e^{tsC}e^{sA}$ for all $t\geqq 0$. This completes the proof of the theorem.

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References

[1] Foias, C., L. Geher and B. Sz.-Nagy: On the permutability condition of quantum mechanics. Acta Sci. Math. (Szeged), **21**, 78–89 (1960).

[2] T. Kato: On the commutation relation $AB-BA=c$. Arch. for Rat. Mech. and Anal., **10**, 273–275 (1962).
[3] E. Hille and R. S. Phillips: Functional Analysis and Semi-Groups. Amer. Math. Soc. Coll. Publ., Vol. 31 (1957).
[4] H. F. Trotter: On the product of semi-groups of operators. Proc. Amer. Math. Soc., **10**, 545–551 (1959).