39. *Some Properties of wM-Spaces*

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1. Recently the notion of wM-spaces has been introduced by one of the authors (cf. T. Ishii [7]), which is a generalization of the notion of M-spaces due to K. Morita [12]. The purpose of this paper is to state further properties of wM-spaces, most of which are concerned with metrization of wM-spaces. A topological space X is called a wM-space if there exists a sequence $\{\mathfrak{U}_n\}$ of open coverings of X satisfying the condition below:

(M_2) $\begin{cases}\text{If } \{K_n\} \text{ is a decreasing sequence of non-empty subsets of } X \text{ such} \\ \text{that } K_n \subset \mathrm{St}^2(x_0, \mathfrak{U}_n) \text{ for each } n \text{ and for some fixed point } x_0 \text{ of } X, \\ \text{then } \cap \bar{K}_n \neq \emptyset.\end{cases}$

In the above definition, we may assume without loss of generality that $\{\mathfrak{U}_n\}$ is decreasing.

Now we shall state a remarkable property of wM-spaces, which was essentially proved in [7, I, Theorem 2.4].

Theorem 1.1. *Let X be a wM-space, and $\{F_\lambda\}$ a locally finite collection of closed subsets of X. Then there exists a locally finite collection $\{H_\lambda\}$ of open subsets of X such that $F_\lambda \subset H_\lambda$ for each λ.*

This theorem will play an important role in the proof of the following theorems. For topological spaces no separation axiom is assumed unless otherwise specified.

2. First we shall prove the following

Theorem 2.1. *Every point-finite open covering of a wM-space has a locally finite open refinement.*

This theorem is an immediate consequence of the following lemmas.

Lemma 2.2. *Every point-finite open covering of a wM-space has a σ-locally finite open refinement.*

With the aid of Theorem 1.1, we can prove this lemma by the similar method as in the case of a collectionwise normal space (cf. E. Michael [9], K. Nagami [14]), and hence we omit the proof.

Lemma 2.3 ([7, I, Lemm 2.6]). *Every wM-space is countably paracompact.*

3. Recently the interesting properties of semi-stratifiable spaces has been obtained by G. D. Creede [3] and Ja. A. Kofner [8].

Theorem 3.1. *A space X is metrizable if and only if X is a T_2, semi-stratifiable wM-space.*

Proof. We shall only prove the 'if'-part, since the 'only-if'-part is obvious. Let X be a T_2, semi-stratifiable wM-space. Since a semi-stratifiable space is F_σ-screenable, that is, every open covering has a σ-discrete closed refinement (cf. [3], [8]), any open covering of X has a σ-locally finite open refinement by Theorem 1.1. Hence, by Lemma 2.3, X is paracompact. On the other hand, X has a G_δ-diagonal, that is, the diagonal Δ of the product $X\times X$ is a G_δ-set. In fact, the set Δ is closed in $X\times X$, since X is T_2, and the product $X\times X$ is semi-stratifiable (cf. [3], [8]). Therefore it follows that Δ is a G_δ-set in $X\times X$. Hence by a theorem of A. Okuyama [17, Theorem 1'] X is metrizable. Thus we complete the proof.

A space X has a $G_\delta(k)(\bar{G}_\delta(k))$-diagonal, $k=1,2,\cdots$, if there exists a sequence $\{\mathfrak{V}_n\}$ of open coverings of X such that for distinct points x,y there exists a positive integer m satisfying $y\notin \mathrm{St}^k(x,\mathfrak{V}_m)(y\notin \overline{\mathrm{St}^k(x,\mathfrak{V}_m)})$, where we may assume that $\{\mathfrak{V}_n\}$ is decreasing.

The following theorem is an improvement of [7, II, Theorems 2.1 and 2.2], and inculudes a metrization theorem due to K. Morita [11, Theorem 4], A. H. Stone [18, Theorem 1] and A. Arhangel'skii [1, Theorem 2].

Theorem 3.2. *A space X is metrizable if and only if X is a wM-space with a $\bar{G}_\delta(1)$-diagonal.*

Proof. We shall only prove the 'if'-part. Let X be a wM-space with a $\bar{G}_\delta(1)$-diagonal. Then X is obviously T_2. Let $\{\mathfrak{U}_n\}$ be a decreasing sequence of open coverings of X satisfying (M_2), and $\{\mathfrak{V}_n\}$ a decreasing sequence of open coverings of X such that for distinct points x,y there exists a positive integer m satisfying $y\notin\overline{\mathrm{St}(x,\mathfrak{V}_m)}$. If we put $\mathfrak{W}_n=\mathfrak{U}_n \cap\mathfrak{V}_n$, $n=1,2,\cdots$, then it is easily proved that $\{\mathrm{St}(x,\mathfrak{W}_n)|n=1,2,\cdots\}$ is a local base at each point x of X. Hence it follows that X is semi-stratifiable. Therefore, by Theorem 3.1, X is metrizable. Thus we complete the proof.

Since a semi-metric space is semi-stratifiable, it follows from Theorem 3.1 that a T_2, semi-metric wM-space is metrizable. Further, we can prove the following theorem, which is a generalization of a theorem of S. Nedev [16, Theorem 4] for M-spaces.

Theorem 3.3. *A space X is metrizable if and only if X is a T_2, symmetrizable wM-space.*

Proof. We shall only prove the 'if'-part. Let $\{\mathfrak{U}_n\}$ be a decreasing sequence of open coverings of X satisfying (M_2). First we show that every one-point set $\{x\}$ of X is a G_δ-set in X. For this purpose, let us put $C(x)=\cap\mathrm{St}(x,\mathfrak{U}_n)$ and $D(x)=\cap\overline{\mathrm{St}(x,\mathfrak{U}_n)}$. Then $D(x)$ is countably compact by (M_2), and it is symmetrizable as a closed subspace of X. Since a T_2, countably compact and symmetrizable space is compact (cf. [16]), $D(x)$ is metrizable by a theorem of A. Arhangel'skii [2, Proposition 2]. Hence it follows that $C(x)$ is metrizable. Let $\{N_n(x)\}$ be a local open base at x in $C(x)$. Then for each n there is an open neighborhood $V_n(x)$ of x in X such that $N_n(x)=V_n(x)\cap C(x)$. Therefore we have $\{x\}=\cap N_n(x)=\cap(V_n(x)\cap C(x))=(\cap V_n(x))\cap(\cap \mathrm{St}(x,\mathfrak{U}_n))$, which shows that $\{x\}$ is a G_δ-set in X. Next we show that X is regular. Let U be an open neighborhood of a point x of X. If $D(x)\subset U$, then it holds that $D(x)\subset\overline{\mathrm{St}(x,\mathfrak{U}_n)}\subset U$ for some n. If $D(x)\cap(X-U)\neq\emptyset$, then there is an open set Q such that $D(x)\cap(X-U)\subset Q$ and $x\notin\bar{Q}$, since X is T_2 and $D(x)\cap(X-U)$ is compact. Further, since $D(x)\subset U\cup Q$, it follows that $\overline{\mathrm{St}(x,\mathfrak{U}_n)}\subset U\cup Q$ for some n. Hence, if we put $V(x)=\mathrm{St}(x,\mathfrak{U}_n)\cap(X-\bar{Q})$, then we have $\overline{V(x)}\subset\overline{\mathrm{St}(x,\mathfrak{U}_n)}\cap(X-Q)\subset U$. Thus X is regular. Finally we show that X satisfies the first countability axiom. Since each one-point set $\{x\}$ is a G_δ-set in X, there is a decreasing sequence $\{U_n(x)\}$ of open sets in X such that $\{x\}=\cap U_n(x)$. Further, since X is regular, there is a dereasing sequence $\{V_n(x)\}$ of open sets in X such that $x\in V_n(x)\subset\overline{V_n(x)}\subset U_n(x)$ for each n. Let us put $W_n(x)=V_n(x)\cap\mathrm{St}(x,\mathfrak{U}_n)$. Then it is easily verified that $\{W_n(x)\}$ is a local base at x. Therefore X satisfies the first countability axiom, which implies that X is semi-metric, since X is symmetrizable. Hence X is metrizable. Thus we complete the proof.

4. As a generalization of the theorems obtained by V. V. Filippov [4] and J. Nagata [15], J. Suzuki has recently proved the following: A space X is metrizable if and only if X is a regular, T_1, M-space with a point-countable collection $\mathfrak{U}$ of open subsets of X such that $\{x\}=\cap\{\bar{U}\mid x\in U, U\in\mathfrak{U}\}$ for each point x of X. The following theorem includes Suzuki's result as a corollary.

Theorem 4.1. *A space X is metrizable if and only if X is a wM-space with the sequence $\{\mathfrak{U}_n\}$ of σ-point-finite open coverings of X satisfying* (M_2) *and with a point-countable collection $\mathfrak{V}$ of open subsets of X such that $\{x\}=\cap\{\bar{V}\mid x\in V, V\in\mathfrak{V}\}$ for each point x of X.*[1)]

The proof is a slight modification of Filippov's and Nagata's. First we mention some lemmas.

Lemma 4.2 (A. S. Miščenko [10]). *Let $\mathfrak{V}$ be a point-countable collection of subsets of a set X and X' a subset of X. Then there are at most countably many finite minimal coverings of X' by elements of $\mathfrak{V}$, where we mean by a minimal covering a covering which contains no proper subcover.*

Lemma 4.3. *Let X be a countably compact space and $\mathfrak{V}$ a point-countable open pseudobase, that is, $\{x\}=\cap\{V\mid x\in V, V\in\mathfrak{V}\}$ for each*

1) A collection $\mathfrak{U}$ of subsets of a space X is said to be σ-point-finite if $\mathfrak{U}=\cup\mathfrak{U}_n$, where $\mathfrak{U}_n$ is a point-finite collection of subsets of X.

point x of X. Then $\mathfrak{V}$ is countable.[2]

Proof. Let Φ be the set of all finite minimal coverings of X by elements of $\mathfrak{V}$. Then by Lemma 4.2 we have $|\Phi|\leqq\aleph_0$, where $|\Phi|$ is the cardinal number of the set Φ. Therefore it is sufficient to show that any element V_0 of $\mathfrak{V}$ belongs to some ω in Φ, where we may assume that $X-V_0\neq\emptyset$. Let $x_0\in V_0$ and $x_1\in X-V_0$, and let us put $\mathfrak{V}_1=\{V\,|\,x_1\in V, x_0\notin V, V\in\mathfrak{V}\}$. Then $\mathfrak{V}_1\neq\emptyset$ and $|\mathfrak{V}_1|\leqq\aleph_0$. In case $\mathfrak{V}_0\cup\mathfrak{V}_1$ covers X, where $\mathfrak{V}_0=\{V_0\}$, it follows that V_0 belongs to some ω in Φ. In case $\mathfrak{V}_0\cup\mathfrak{V}_1$ does not cover X, we take a point x_2 not belonging to the union of sets of $\mathfrak{V}_0\cup\mathfrak{V}_1$ and put $\mathfrak{V}_2=\{V\,|\,x_2\in V, x_0\notin V, V\in\mathfrak{V}\}$. Then $\mathfrak{V}_2\neq\emptyset$ and $|\mathfrak{V}_2|\leqq\aleph_0$. In case $\mathfrak{V}_0\cup\mathfrak{V}_1\cup\mathfrak{V}_2$ does not cover X, we repeat the above procedure. Then it is proved that $\mathfrak{V}_0\cup\mathfrak{V}_1\cup\cdots\cup\mathfrak{V}_n$ covers X for some n. Indeed, otherwise, there exists an infinite subset $\{x_n\}$ of X such that $\mathfrak{V}_0\cup\mathfrak{V}_1\cup\cdots\cup\mathfrak{V}_{n-1}$ does not contain x_n for each n. Since X is countably compact, there is an accumulation point y_0 of the set $\{x_n\}$. Let V be an element of $\mathfrak{V}$ such that $y_0\in V$ and $x_0\notin V$. Since V contains infinitely many elements of $\{x_n\}$, we can take two points $x_i, x_j (i<j)$ in V. Then we have $V\in\mathfrak{V}_i$. But x_j does not belong to the union of sets of $\mathfrak{V}_0\cup\mathfrak{V}_1\cup\cdots\cup\mathfrak{V}_i$ in view of the above process. This is a contradiction. Since $\mathfrak{V}_0\cup\mathfrak{V}_1\cup\cdots\cup\mathfrak{V}_n$ is a countable open covering of X, it has a finite minimal subcovering to which V_0 belongs. Thus we complete the proof.

Lemma 4.4. *A space X is metrizable if and only if X is a T_2, wM-space with a σ-point-finite open base.*

Proof. We shall only prove the 'if'-part. Let X be a T_2, wM-space with a σ-point-finite open base. Then by a theorem of S. Hanai [6, Theorem 1] X is developable, and hence X is semi-stratifiable. Therefore X is metrizable by Theorem 3.1. Thus we complete the proof.

Remark. In [6], S. Hanai pointed out that a T_1-space X has a σ-point-finite open base if and only if X has a uniform base. Hence Lemma 4.4 shows that a T_2, wM-space with a uniform base is metrizable.

Proof of Theorem 4.1. We may assume that $X\in\mathfrak{V}$. Let $U\in\mathfrak{U}_n$. Then by Lemma 4.2 there are at most countably many finite minimal coverings of U by members of $\mathfrak{V}$, which we denote by $\mathfrak{W}_m(U), m=1,2,\cdots$ (In case there are only finitely many of them, we repeatedly count a cover). Let $\mathfrak{W}'_m(U)$ be the restriction of $\mathfrak{W}_m(U)$ to U, and let us put $\mathfrak{W}_{m,n}=\cup\{\mathfrak{W}'_m(U)|\,U\in\mathfrak{U}_n\}$ for each m and n. Then $\mathfrak{W}_{m,n}$ is a σ-point-finite collection of open subsets of X. To show that $\cup\mathfrak{W}_{m,n}$ is an open

2) J. Suzuki proved Lemma 4.3 under the condition that $\{x\}=\cap\{\overline{V}|x\in V, V\in\mathfrak{V}\}$ for each point x of X in place of the pseudobase.

base of X, let $x_0 \in X$ and N an open neighborhood of x_0 Let us put $D(x_0) = \cap \overline{\mathrm{St}(x_0, \mathfrak{U}_n)}$. If $D(x_0) \subset N$, then there is nothing to prove. Suppose that $D(x_0) - N \neq \emptyset$. Then it is obvious that $D(x_0) - N$ is countably compact. Hence there are $V'_1, \cdots, V'_k \in \mathfrak{V}$ such that $x_0 \in V'_1 \cap \cdots \cap V'_k \subset \bar{V}'_1 \cap \cdots \cap \bar{V}'_k \subset X - (D(x_0) - N)$. Indeed, otherwise, the countable family $\mathfrak{F} = \{(D(x_0) - N) \cap \bar{V} \mid x_0 \in V, V \in \mathfrak{V}\}$ of closed subsets of X has a finite intersection property, and hence the intersection of all elements of $\mathfrak{F}$ is not empty. If we take a point y belonging to this intersection, then there is some V of $\mathfrak{V}$ such that $x_0 \in V$ and $y \notin \bar{V}$, which is a contradiction. Let us put $V_0 = V'_1 \cap \cdots \cap V'_k$. Since we may assume without loss of generality that the intersection of any finitely many members of $\mathfrak{V}$ belongs to $\mathfrak{V}$, V_0 is a member of $\mathfrak{V}$ satisfying $x_0 \in V_0 \subset \bar{V}_0 \subset X - (D(x_0) - N)$. Further, by Lemma 4.3, there are $V_1, \cdots, V_l \in \mathfrak{V}$ such that $D(x_0) - V_0 \subset V_1 \cup \cdots \cup V_l$ and $x_0 \notin \bigcup_{i=1}^{l} V_i$, where we may assume that $\{V_0, V_1, \cdots, V_l\}$ is a minimal covering of $D(x_0)$. Let us put $W = V_0 \cup V_1 \cup \cdots \cup V_l - (\bar{V}_0 - N)$. Then W is an open set in X such that $D(x_0) \subset W$, since $\bar{V}_0 \cap (D(x_0) - N) = \emptyset$. Hence we have $D(x_0) \subset \overline{\mathrm{St}(x_0, \mathfrak{U}_n)} \subset W$ for some n. Let U_0 be an element of $\mathfrak{U}_n$ containing x_0. Then, since $U_0 \subset W \subset V_0 \cup V_1 \cup \cdots \cup V_l$ and $x_0 \notin V_1 \cup \cdots \cup V_l$, there is a finite minimal covering $\{V_0, V_{i_1}, \cdots, V_{i_\mu}\}$ of U_0, where $V_{i_1}, \cdots, V_{i_\mu}$ are the members of $\{V_1, \cdots, V_l\}$. Since $\{V_0, V_{i_1}, \cdots, V_{i_\mu}\} = \mathfrak{W}_m(U_0)$ for some m, we have $U_0 \cap V_0 \in \mathfrak{W}'_m(U_0) \subset \mathfrak{W}_{m,n}$. Further, from $U_0 \subset W$ and $W \cap V_0 \subset W \cap \bar{V}_0 \subset N$, it follows that $U_0 \cap V_0 \subset N$. Therefore $\cup \mathfrak{W}_{m,n}$ is a σ-point-finite open base of X. Thus X is metrizable by Lemma 4.4, since X is obviously T_2 by the assumption. This completes the proof.

Corollary 4.5. *A space X is metrizable if and only if X is an, wM-space with the sequence $\{\mathfrak{U}_n\}$ of σ-point-finite open coverings of X satisfying* (M$_2$) *and with a point-countable base.*

Corollary 4.6. *A space X is metrizable if and only if X is an M-space with a point-countable open collection $\mathfrak{V}$ such that $\{x\} = \cap \{\mathfrak{V} \mid x \in V, V \in \mathfrak{V}\}$ for each point x of X.*

In connection with Theorem 4.1, we note that a locally compact, T_2, wM-space with the sequence $\{\mathfrak{U}_n\}$ of locally finite open coverings of X satisfying (M$_2$) is not an M-space in general; this is easily proved by an example of K. Morita [13].

5. Finally we mention a theorem which is a slight modification of Hanai's theorem [6] for a collectionwise normal space.

Theorem 5.1. *If X is a T_2, wM-space and $X = \bigcup_{n=1}^{\infty} G_n$, where each G_n is an open metrizable subset, then X is metrizable.*

Since this theorem can be easily proved with the aid of Lemma 4.4, we omit the proof.

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