34. *Construction of a Local Elementary Solution for Linear Partial Differential Operators.* II

By Takahiro KAWAI
Research Institute for Mathematical Sciences, Kyoto University

(Comm. by Kunihiko KODAIRA, M. J. A., Feb. 12, 1971)

Let $P(x, D_x)$ be a linear partial differential operator with real analytic coefficients defined on a domain containing the origin in $\boldsymbol{R}^n$. We denote its principal symbol by $P_m(x, \xi)$. Assume that $P(x, D_x)$ has simple characteristics, that is, $\text{grad}_\xi P_m(x, \xi) \neq 0$ whenever $P_m(x, \xi) = 0$.

In this note we first construct a local elementary solution for P under the condition (P), which is concerned with the behaviour of the characteristic surfaces. Secondly we prove that the condition (P) follows from the condition $(\text{NT})_f$, which is deeply related with the work of Nirenberg and Treves [6], [7]. The condition $(\text{NT})_f$ does not cover all the possibilities of the solvable partial differential operators in the theory of hyperfunctions. Thus our result is weaker than that of Nirenberg and Treves [7] concerning distribution solutions. Our analysis is different from theirs in the point that we treat the problem in the framework of hyperfunctions or rather in that of Sato's sheaf $\mathcal{C}$ defined on the cotangential sphere bundle (or co-sphere bundle in short). For the notion of the sheaf $\mathcal{C}$ we refer the reader to Sato [8], [9]. We hope, however, our method of construction of an elementary solution given in Theorem 2 reveals the geometrical meaning of condition $(\text{NT})_f$.

In Theorem 4 and Theorem 5 we also treat two cases which are not covered by condition $(\text{NT})_f$. We remark that the three features, which appear in Theorems 2, 4 and 5 respectively, are typical ones about the behaviour of the characteristic surfaces.

We have constructed a local elementary solution $E(x, y)$ for a linear partial differential operator P with simple characteristics and with real coefficients in its principal symbol and investigated its singularities in our previous note [4], so that in the sequel we consider the case where the principal symbol $P_m(x, \xi)$ of P has the form $A_m(x, \xi) + iB_m(x, \xi)$ where A_m and B_m are real and $B_m \not\equiv 0$. We can assume that $\text{grad}_\xi A_m \neq 0$ when $P_m = 0$ without the loss of generalities. The details of this note will be published elsewhere. (See also Kawai [5].)

Our method of construction of an elementary solution for P is just the same as that employed in our previous note [4]. We first repeat the fundamental theorem essentially due to Hamada [1] in a form which is suitable for the present situations.

Let $P(z, D_z)$ be a linear partial differential operator with holomorphic coefficients defined near the origin of C^n. Assume that $P(z, D_z)$ has the form $\sum_{j=0}^{m} a_j(z, D_{z'})\partial^{j-1}/\partial z_1^{j-1}$ where z' denotes $(z_2, \cdots, z_n)$, that the order of $a_j(z, D_{z'})$ is equal to or smaller than $m-j$ and that $a_m(z, D_{z'}) \equiv 1$. Assume further that $\partial/\partial\xi_1 P_m(z, \xi) \neq 0$ near $(z, \xi)=(0, \xi^0)$ where $P_m(0, \xi^0)=0$. Then we have the following Theorem 1. In this theorem we denote by $\chi(z, w, \xi)$ a holomorphic function in (z, w, ξ) near $(0, 0, \xi^0)$, positively homogeneous of degree 1 with respect to ξ and has the form $\langle z-w, \xi\rangle + O(|z-w|^2|\xi|)$. This function χ is specified later in the course of the construction of elementary solution to expand $\delta(x-y)$ using curvilinear waves. (The expansion of δ-function by complex-valued curvilinear waves is due to Sato.)

Theorem 1. *Consider the following singular Cauchy problem:*
$$\begin{cases} P(z, D_z)E(z, w, \xi, s)=0 \\ P'(z, D_z, D_s)E(z, w, \xi, s)|_{z_1=s}=J(z, w, \xi)/\chi(z, w, \xi)^n, \end{cases}$$
where $J(z, w, \xi)$ is a holomorphic function in (z, w, ξ) and $P'(z, D_z, D_s)$ is defined by giving its symbol as $P(z, \sigma+\xi_1, \xi')-P(z, \xi)/\sigma$, where σ stands for D_s. Then we have a solution $E(z, w, \xi, s)$, which is represented in the form $\sum_{j=m-n-1}^{-1} e_j(z,w,\xi,s)\varphi^j(z,w,\xi,s)+e_0(z,w,\xi,s)\log\varphi(z,w,\xi,s)+e_1(z,w,\xi,s)$, where e_j's are holomophic functions and $\varphi(z, w, \xi, s)$ satisfies the characteristic equation $P_m(z, \mathrm{grad}_z\,\varphi(z, w, \xi, s))=0$ with the initial data $\chi(z, w, \xi)$ on $\{z_1=s\}$.

Erratum. In our previous note [4], the operator $P'(z, D_z)$ defined in Theorem 1 should be replaced by the above $P'(z, D_z, D_s)$.

We next give the definition of condition $(\mathrm{P})_{(0,\xi^0)}$. In the sequel we drop the subscript $(0, \xi^0)$ for convenience.

Condition (P): Choosing a suitable initial condition $\chi(z, w, \xi)$ which is positively homogeneous of degree 1 with respect to ξ and for which $\mathrm{Im}\,\chi(x, y, \xi)\geqq 0$ holds on $\{(x, y, \xi)$ real and $\mathrm{Re}\,\chi(x, y, \xi)=0\}$, we have $\mathrm{Im}\,\varphi(x, y, \xi, s)\geqq 0$ on $\{(x, y, \xi, s)$ real, $x_1\geqq s, \xi\in I^+$ and $\mathrm{Re}\,\varphi(x, y, \xi, s)=0\}$ and $\{(x, y, \xi, s)$ real, $x_1\leqq s, \xi\in I^-$ and $\mathrm{Re}\,\varphi(x, y, \xi, s)=0\}$ where I^+ and I^- are locally closed set in an $(n-1)$-dimensional co-sphere S^{n-1} and their union $I=I^+\cup I^-$ is a neighbourhood of ξ^0 in S^{n-1}.

Remark 1. When the space dimension n is larger than 2 the suitable choice of $\chi(z, w, \xi)$ is important since we cannot solve the Hamilton-Jacobi equations in a real domain in general to obtain $\varphi(x, y, \xi, s)$ for an operator with complex coefficients.

Remark 2. In general, a real analytic function $f(x)$ is said, after Sato, to be of positive type if $\mathrm{Im}\,f(x)\geqq 0$ holds when $\mathrm{Re}\,f(x)=0$. Analogous to this terminology condition (P) may be said as follows: the phase function φ can be chosen to be of half positive type for a suitable choice of the initial condition χ which is of positive type. The notion of the half positive type is the key to the solvability. Compare the fact that the singularity of a good elementary solution for a operator P with real principal symbol is contained in "half of the bicharacteristic strips". (See Kawai [4] about the precise statement.)

Theorem 2. *Assume that P satisfies condition* (P). *Then we can construct $E(x,y)$ for which $P(x,D_x)E(x,y)=\delta(x-y)$ holds near* $(0,0,\xi^0,-\xi^0)$ *as the section of the sheaf $\mathcal{C}$.*

Sketch of the proof. We first choose $\chi_j(x,y,\xi)$ so that they satisfy $\chi(x,y,\xi)=\sum_{j=1}^n (x_j-y_j)\chi_j(x,y,\xi)$ and are positively homogeneous of degree 1 with respect to ξ. Then we define $J(x,y,\xi)$ by $\dfrac{\partial(\chi_1,\cdots,\chi_n)}{\partial(\xi_1,\cdots,\xi_n)}$ and apply Theorem 1. Finally we define $E(x,y)$ as the boundary value of $\int_{I^+}\omega(\xi)\int_\alpha^{x_1}E(x_1,z',y,\xi,s)ds-\int_{I^-}\omega(\xi)\int_{x_1}^{\beta}E(x_1,z',y,\xi,s)ds$ from the domain $\{\mathrm{Im}\,\varphi>0\}$, where $\omega(\xi)$ is the volume element $\sum_{j=1}^n(-1)^{j-1}\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$ and α and β are some constants. The above integral is well-defined by condition (P). It is obvious from the initial condition for $E(z,w,\xi,s)$ given in Theorem 1 that $P(x,D_x)E(x,y)=\int_I\dfrac{J(x,y,\xi)}{(\chi(x,y,\xi)+i0)^n}\omega(\xi)$ holds. By Sato's formula for the curvilinear wave decomposition of δ-function we have $\int\dfrac{J(x,y,\xi)}{(\chi(x,y,\xi)+i0)^n}\omega(\xi)=\dfrac{(-2\pi i)^n}{(n-1)!}\delta(x-y)$ since χ is of positive type. Thus we have obtained the required $E(x,y)$.

We next investigate the relation between condition (P) and condition $(\mathrm{NT})_f$, which is related to the operator P itself more directly than (P). We denote by $\Gamma_{(x_0,\xi^0)}$ the bicharacteristic strip of $A_m(x,\xi)$ inssuing from (x_0,ξ^0) which satisfies $P_m(x_0,\xi^0)=0$.

Condition $(\mathrm{NT})_f$: $B_m(x,\xi)$ has a zero of finite even order at $(x_0',\xi^{0\prime})$ along $\Gamma_{(x_0',\xi^{0\prime})}$ for $|x_0-x_0'|\ll 1$ and $|\xi^0-\xi^{0\prime}|\ll 1$. (Cf. Nirenberg and Treves [7] p. 460. Their condition $(\mathscr{P})$ admits $B_m\equiv 0$ on some $\Gamma_{(x_0',\xi^{0\prime})}$.)

Theorem 3. *Condition* $(\mathrm{NT})_f$ *implies condition* (P).

Sketch of the proof. It is sufficient to find a suitable initial condition χ so that the phase function φ becomes of half positive type for some suitable choice of local co-ordinate system. Since we have assumed that P has simple characteristics, $P_m(x,\xi)$ can be decomposed into the form $Q(x,\xi)(\xi_1-a(x,\xi')-ib(x,\xi'))$ near $(x,\xi)=(0,\xi^0)$, where $Q(x,\xi)$ never vanishes near $(0,\xi^0)$ and is positively homogeneous of degree $m-1$ with respect to ξ and $a(x,\xi')$ and $b(x,\xi')$ are real valued and positively by homogeneous of degree 1 with respect to $\xi'=(\xi_2,\cdots,\xi_n)$. Using the invariance property of $(\mathrm{NT})_f$ by multiplication of non-vanishing factor due to Nirenberg and Treves [6] § 2, we can assume that $\xi_1-a(x,\xi')-ib(x,\xi')$ satisfies condition $(\mathrm{NT})_f$. Thus it is sufficient to investigate the properties of φ which satisfies $\partial\varphi/\partial x_1-a(x,\mathrm{grad}_{x'}\varphi)-ib(x,\mathrm{grad}_{x'}\varphi)=0$. We have a unique holomorphic solution of the above first order partial differential equation when the initial condition for φ is given on some non-characteristic surface by the integration of the Hamilton-Jacobi equations in a complex domain since we have assumed that the coefficients of $P(x,D_x)$ are real analytic. So we estimate $\mathrm{Im}\,\varphi$ using the asymptotic expansion of φ. After the usual real coordinate transformation from (x) to (y) which straightens the bicharacteristic strip through (x_0,ξ^0), that is, the bicharacteristic strip of $\eta_1-\tilde{a}(y,\eta')$ is parallel to the y_1-axis, where $\eta_1-\tilde{a}(y,\eta')$ is the expression of $\xi_1-a(x,\xi')$ after the above coordinate transformation. See Nirenberg and Treves [6] p. 21 as for the coordinate transformation. For the sake of simplicity we write (x,ξ) instead of (y,η) after the coordinate transformation and use the letter y to denote a parameter as in Theorem 2. It is obvious from our assumption that we have for some neighbourhood V of the origin in $\boldsymbol{R}^n$ $b(x,\xi)\geqq 0$ in V, or $b(x,\xi)\leqq 0$ in V. So we assume that $b(x,\xi)\geqq 0$ in V. Assuming that $\varphi(x,y,\xi,s)$ has the form $(s-y_1)\xi_1+\langle x'-y',\xi'\rangle+i|x'-y'|^2+\sum_{k=0}^{\infty}\varphi_k(x,y,\xi,s)$, where y,ξ and s play the role of parameters, $\varphi_k(x,y,\xi,s)$ are polynomials in $(x'-y')$ of order k and $\varphi_k(s,x',y,\xi,s)=0$ for every k. Then φ_k's are determined successively by solving ordinary differential equations and it is not difficult to estimate $\mathrm{Im}\,\varphi$ for $x_1\geqq s$. (Cf. Nirenberg and Treves [6] pp. 22–25). Thus we conclude that condition (P) follows from $(\mathrm{NT})_f$.

Remark. In the above argument we have proved more than (P) because $\mathrm{Im}\,\varphi>0$ if $x_1>s$ (or $x_1<s$). Hence we hope condition (P) will be satisfied even when $B_m(x,\xi)$ vanishes identically on some bicharacteristic strip of $A_m(x,\xi)$, but we have not yet proved this fact.

As is clear from the above remark the case which condition $(\mathrm{NT})_f$ covers is one extreme case where P has a local elementary solution. There are two other extreme cases which are easily treated by the theory of pseudo-differential operators of finite type developed in Kashiwara and Kawai [3]. Since the method is just the same as that indicated in the last part of our previous note [4] and its idea is due to Hörmander [2], we do not repeat its procedure in this note but indicate where the changes are needed. Until the end of this note we assume that the vectors $\mathrm{grad}_\xi A_m(x,\xi)$ and $\mathrm{grad}_\xi B_m(x,\xi)$ are linearly independent whenever $P_m(x,\xi)=0$. In some cases we may use the assumption of the linear independence of $\mathrm{grad}_{(x,\xi)} A_m$ and $\mathrm{grad}_{(x,\xi)} B_m$ on $\{P_m(x,\xi)=0\}$, but under this weaker assumption we must be more careful in technicalities. Therefore we adopt the above stronger condition of linear independence in this note.

Theorem 4. *Assume that there exists a phase function* $\varphi(x, y, \xi)$ *satisfying the following conditions* (i)$\sim$(iv) *near* $(x, y, \xi)=(0, 0, \xi^0)$. *Then we can construct* $E(x, y)$ *which satisfies* $P(x, D_x)E(x, y)=\delta(x-y)$ *near* $(0, 0, \xi^0, -\xi^0)$ *as sections of the sheaf* $\mathcal{C}$.

(i) $P_m(x, \mathrm{grad}_x \varphi(x, y, \xi))=P_m(y, \xi)$

(ii) $\varphi(x, y, \xi)=\langle x-y, \xi\rangle+O(|x-y|^2|\xi|)$

(iii) $\varphi(x, y, \xi)$ *is real analytic near* $(0, 0, \xi^0)$ *and positively homogeneous of degree* 1 *with respect to* ξ.

(iv) $\varphi(x, y, \xi)$ *is of positive type.*

The method of the construction of $E(x, y)$ given in our previous note [4] Theorem 2′ runs in this case without any essential changes. Remark that $1/P_m(y, \xi)$ is well-defined using the theory of substitutions in the sheaf $\mathcal{C}$ (Sato [9]) since we have assumed $\mathrm{grad}_\xi A_m$ and $\mathrm{grad}_\xi B_m$ are linearly independent when $P_m(x, \xi)=0$.

Remark. The local elementary solution $E(x, y)$ constructed above plays an essential role to characterize the structure of the sheaf $\mathrm{Coker}_{\mathcal{C}} P^*$ using another pseudo-differential operator. The details will be given in our next note.

We denote by $\overline{P_m}(x, \xi)$ the form with complex conjugate coefficients of $P_m(x, \xi)$, that is, $\overline{P_m}(x, \xi)=\sum_{|\alpha|=m} \overline{a_\alpha(x)}\xi^\alpha$ if $P_m(x, \xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

Theorem 5. *Assume that the commutator of* $P_m(x, D_x)$ *and* $\overline{P_m}(x, D_x)$ *vanishes identically. Then we can construct a local elementary solution near* $(0, 0, \xi^0, -\xi^0)$ *for any* ξ^0.

In this case we can integrate the Hamilton-Jacobi equations in a real domain and obtain real valued $\varphi(x, y, \xi)$ satisfying $P_m(x, \mathrm{grad}_x \varphi)=P_m(y, \xi)$ near $(0, 0, \xi^0)$ and positively homogeneous of degree 1 with respect to ξ. Thus the proof is just the same as in our previous note [4] Theorem 2′.

Remark. If $P_m(x, y, D_x, D_y)$ has the form $Q_m(x+iy, D_x-iD_y)$ for some $Q_m(z, \zeta)$, then the condition of Theorem 5 is trivially satisfied. Therefore such an operator is very close to an operator with real principal symbol from the viewpoint of the behaviour of the characteristic surfaces. Such a class of operators appeared in a discussion with Sato and Kashiwara.

References

[1] Hamada, Y.: The singularities of the solutions of the Cauchy problem. Publ. R.I.M.S. Kyoto Univ. Ser. A, **5**, 21–40 (1969).

[2] Hörmander, L.: On the singularity of solutions of partial differential equations. Proc. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, pp. 31–40 (1969).

[3] Kashiwara, M., and T. Kawai: Pseudo-differential operators in the theory of hyperfunctions. Proc. Japan Acad., **46**, 1130–1134 (1970).

[4] Kawai, T.: Construction of a local elementary solution for linear partial differential operators. I. Proc. Japan Acad., **47**, 19–23 (1971).
[5] ——: On the local theory of (pseudo)-differential operators. Reports of the Symposium on the Theory of Hyperfunctions. R. I. M. S. Kyoto University, pp. 1–45 (1970) (in Japanese).
[6] Nirenberg, L., and F. Treves: On local solvability of linear partial differential equations, Part I. Comm. Pure Appl. Math., **23**, 1–38 (1970).
[7] ——: On local solvability of linear partial differential equations, Part II. Ibid., **23**, 459–510 (1970).
[8] Sato, M.: Hyperfunctions and partial differential equations. Proc. Conf. on Functional Analysis and Related Topics, Univ. of Tokyo Press, pp. 91–94 (1969).
[9] ——: On the structure of hyperfunctions. Sûgaku no Ayumi, **15**, 9–72 (1970) (Notes by Kashiwara, in Japanese).