33. *On Semigroups whose Ideals are All Globally Idempotent*

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A left (right, two-sided) ideal X of a semigroup S is called globally idempotent if $X^2=X$ (according to S. Lajos). As is well-known, a commutative semigroup is regular if and only if every ideal is globally idempotent ([3]), and a normal semigroup is regular if and only if every left ideal is grobally idempotent ([5]). J. Calais characterized semigroups whose right ideals or left ideals are all globally idempotent ([2]). Recently R. C. Courter [2] has given an interesting characterisation of rings whose ideals are all globally idempotent. In this note we give a characterisation for a semigroup which is similar to Theorem 1.2 of R. C. Courter [2]. For another properties of semigroups whose ideals are all globally idempotent see [4] and [6].

Let X and Y be subsets of a semigroup S. Then we put

$$(Y:X)=\{s\in S: xs\in Y \quad \text{for all } x\in X\}$$
$$(Y:X)^*=\{s\in S: sx\in Y \quad \text{for all } x\in X\}.$$

Theorem 1. *The following statements on a semigroup S are equivalent*:

(1) $X^2=X$ *for every ideal X of S.*

(2) $X\cap Y=XY$ *for every ideals X and Y of S.*

(3) $(Y:X)\cap X=X\cap Y$ *for every ideals X and Y of S.*

(4) $(Y:X)^*\cap X=X\cap Y$ *for every ideals X and Y of S.*

(5) $(Y:X)\cap Z=Y\cap Z$ *for every ideals X, Y and Z of S such that $Z\subseteq X$.*

(6) $(Y:X)^*\cap Z=Y\cap Z$ *for every ideals X, Y and Z of S such that $Z\subseteq X$.*

(7) $R\cap X\subseteq XR$ *for every right ideal R and every ideal X of S.*

(8) $L\cap X\subseteq LX$ *for every left ideal L and every ideal X of S.*

(9) $R\subseteq XR$ *for every right ideal R and every ideal X of S such that $R\subseteq X$.*

(10) $L\subseteq LX$ *for every left ideal L and every ideal X of S such that $L\subseteq X$.*

(11) $(R:X)\cap X\subseteq X\cap R$ *for every right ideal R and every ideal X of S.*

(12) $(L:X)^*\cap X\subseteq L\cap X$ *for every left ideal L and every ideal X of S.*

(13) $(R:X)\cap X\subseteq R$ *for every right ideal* R *and every ideal* X *of* S *such that* $R\subseteq X$.

(14) $(L:X)^*\cap X\subseteq L$ *for every left ideal* L *and every ideal* X *of* S *such that* $L\subseteq X$.

Proof. From (2), (1) is obtained by setting $Y=X$. If (1) is assumed and if X and Y are ideals of S, then

$$XY\subseteq X\cap Y=(X\cap Y)^2\subseteq XY.$$

Hence (2) is implied by (1).

We have that (3) implies (2). If X and Y are ideals of S, we have, using (3),

$$Y\cap X\subseteq (XY:X)\cap X=XY\cap X=XY.$$

Hence we obtain (2).

Next if (2) holds and if X and Y are ideals of S, we have, since $(Y:X)$, if it is non-empty, is an ideal of S,

$$(Y:X)\cap X=X(Y:X)\subseteq X\cap Y.$$

The converse inclusion is trivial and (3) has been obtained by (2). (3) is obtained from (5) by setting $Z=X$.

Conversely, if (3) holds and if X, Y and $Z\subseteq X$ are ideals of S, then we have

$$(Y:X)\cap Z\subseteq Z\cap (Y:X)\cap X=Z\cap X\cap Y=Y\cap Z.$$

The converse inclusion is trivial and (5) holds.

Considering the symmetry of (3) and (4), (5) and (6), we have: statements (1) through (6) have been proved equivalent.

(1) is obtained from (9) by setting $R=X$. Thus (9) implies statements (1) through (6).

If the first six statements hold, then by (3) we have for any ideal X of S and any right ideal R of S,

$$X\cap R\subseteq (XR:X)\cap X=XR\cap X=XR.$$

Thus (7) holds. Since (7) clearly implies (9), considering the symmetry of (7) and (8), (9) and (10), it follows that statements (1) through (10) are equivalent.

If (13) is assumed, then we have $XR\subseteq X$ for any right ideal R and any ideal X of S such that $R\subseteq X$, so that

$$R=R\cap X\subseteq (XR:X)\cap X\subseteq XR.$$

Therefore (9) holds.

Conversely let the first ten statements hold. Then for any right ideal R and any ideal X of S, we have by (7)

$$(R:X)\cap X\subseteq X(R:X)\subseteq R\cap X.$$

Thus (11) is implied by any of ten preceding statements.

Clearly, (11) implies (13).

Considering the symmetry of (11) and (12), (13) and (14), the fourteen statements have been proved equivalent. This completes the proof of Theorem 1.

We denote by $[x]_L$($[x]_R$, $[x]$) the principal left (right, two-sided) ideal of S generated by x of S.

Theorem 2. *For a semigroup S, the statements* (A)–(N) *are equivalent with each other and with any of the conditions* (1)–(14) *of Theorem* 1:

(A) $[x]^2=[x]$ *for every element x of S.*

(B) $[x]\cap[y]=[x][y]$ *for every elements x, y of S.*

(C) $([y]:[x])\cap[x]=[x]\cap[y]$ *for every elements x, y of S.*

(D) $([y]:[x])^*\cap[x]=[x]\cap[y]$ *for every elements x, y of S.*

(E) $([y]:[x])\cap[z]=[y]\cap[z]$ *for every elements x, y and z of S such that* $[z]\subseteq[x]$.

(F) $([y]:[x])^*\cap[z]=[y]\cap[z]$ *for every elements x, y and z of S such that* $[z]\subseteq[x]$.

(G) $[a]_R\cap[x]\subseteq[x][a]_R$ *for every elements x, a of S.*

(H) $[b]_L\cap[x]\subseteq[b]_L[x]$ *for every elements x, b of S.*

(I) $[a]_R\subseteq[x][a]_R$ *for every elements x, a of S such that* $[a]_R\subseteq[x]$.

(J) $[b]_L\subseteq[b]_L[x]$ *for every elements x, b of S such that* $[b]_L\subseteq[x]$.

(K) $([a]_R:[x])\cap[x]\subseteq[x]\cap[a]_R$ *for every elements x, a of S.*

(L) $([b]_L:[x])^*\cap[x]\subseteq[b]_L\cap[x]$ *for every elements x, b of S.*

(M) $([a]_R:[x])\cap[x]\subseteq[a]_R$ *for every elements x, a of S such that* $[a]_R\subseteq[x]$.

(N) $([b]_L:[x])^*\cap[x]\subseteq[b]_L$ *for every elements x, b of S such that* $[b]_L\subseteq[x]$.

Proof. It can be seen, in a similar way as in the proof of Theorem 1, that the statements (A)–(N) are equivalent with each other. The condition (1) of Theorem 1 implies the condition (A) of Theorem 2 trivially. Conversely, assume that the condition (A) of Theorem 2 holds, and let x be arbitrary element of any ideal X of S. Since $[x]\subseteq X$, we have

$$x\in[x]=[x]^2\subseteq X^2,$$

and so we have $X\subseteq X^2$. Hence the condition (A) of Theorem 2 implies the condition (1) of Theorem 1. This completes the proof of Theorem 2.

Remark 1. The statements (A)–(B) of Theorem 2 were pointed out to me by Prof. S. Lajos.

Remark 2. For a ring S, the statements (A)–(N) of Theorem 2 are equivalent with each other and with any of the conditions (A)–(P) of Theorem 1.2 of R. C. Courter [2] can be seen in a similar way.

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