30. On Dedekindian l-Semigroups and its Lattice-Ideals

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Our main purpose of the present note is to study some lattice-ideals of Dedekindian l-semigroups. The notation and terminology are those of [1].

1. Let S be an Artinian l-semigroup considered in [1]. An integral element q of S is called primary if the conditions $xy \leq q$, $x \not\leq q$ (x, $y \in I_G$) imply $y^\rho \leq q$ for some positive integer ρ. Then it can be proved that $p \equiv \sup\{x \in I_G \mid x^\rho \leq q$ for some positive integer $\rho\}$ is a prime element in I. Now let $\mathfrak{V} = \{v\}$ be a system of valuations with the properties (A), (B) and (C) in [1]. Then for any fixed $v \in \mathfrak{V}$ and for a primary element q with $q \leq p(v)$ (cf. [1; § 4]), we have that $v(q) \neq 0$ and $v'(q) = 0$ for every $v' \in \mathfrak{V}$ with $v' \neq v$. By using the above fact and the results in [1; § 4], we can prove that, if p is a low prime element of I, the set of the minimal primes less than p consists of infinite many members.

Let p be prime and be not low in I. If we take a valuation $v \in \mathfrak{V}$ such that $v(p) > 0$, then since $v(p) \geq v(p(v)) = 1$, we have $p(v) \geq p$. Now we suppose that $v(p(v)) < v(p)$. Let z be an element such that $z < p$, $z \in I_G$ and $v(z) = v(p)$, and let u be an element such that $u \leq p(v)$, $u \in I_G$ and $v(u) = 1$. Then we can take an element u' such as $zu^{-v(p)}u' = u_0 \leq e$ and $v(u_0) = 0$. By using this and the property "$p(v_1) \neq p(v_2)$ for $v_1 \neq v_2$ in $\mathfrak{V}$", we can show that there exists one and only one valuation v such that $p(v) = p$, $v \in \mathfrak{V}$.

An Artinian l-semigroup is called *Dedekindian* if it has no low element different from e. Then we obtain that any Dedekindian l-semigroup S forms an l-group, and every element a of S is factored into a product of a finite number of primes $p(v)$: $a = \prod_{v \in \mathfrak{V}} p(v)^{v(a)}$, and the factorization is uniquely determined apart from its commutativity. In other words S is the restricted direct product of the cyclic groups $\{p(v)\}$, $v \in \mathfrak{V}$. Now let S be an Artinian l-semigroup. Then the following three conditions are equivalent:

(1) *S is Dedekindian.*

(2) *Each minimal prime of I is maximal.*

(3) *Any two distinct minimal primes are coprime.*

Ad (1)$\Rightarrow$(2): Let p be a minimal prime of I. Then p is written as $p = p(v)$ for some $v \in \mathfrak{V}$. Suppose that there exists an element a such as $p < a \leq e$. Then $v(p) > v(a)$ and $0 \leq v'(a) \leq v'(p) = 0$ for $v' \neq v$, $v' \in \mathfrak{V}$. This implies $a=e$. p is therefore maximal. (2)$\Rightarrow$(3) is evident. Ad (3)$\Rightarrow$(1): Suppose that there exists a low element c with $c\neq e$. Then we can prove the existence of a prime low element p such that $p<e$, and p contains an infinite number of minimal primes. This shows the existence of distinct minimal primes which are not coprime.

2. Let S be a Dedekindian l-semigroup. A lattice ideal J of S is called an *I-lattice ideal* (abv. *I-l-ideal*) of S if $a\in I$ and $c\in J$ imply $ac\in J$. If J is contained in I, J is called *integral.* The main purpose of this section is to determine the I-l-ideals of Dedekindian l-semigroups (with zero).

Now we consider a map $\sigma: p\to p^{\sigma}$ from $\mathfrak{P}$ into $\{\boldsymbol{Z}, -\infty\}$ such that $\{p^{\sigma}\mid p\in\mathfrak{P}\}$ is almost all non-positive, where $\mathfrak{P}$ is the primes in I. Then the set $J(\sigma)$ consisting of the elements c with $v_p(c)\geq p^{\sigma}$ for all $p\in\mathfrak{P}$ forms an I-l-ideal of S.

Conversely we can prove that *any I-l-ideal J of S can be represented as $J=J(\sigma)$ for a suitable map σ.* In the following we shall sketch the proof. First we shall show that $v_{p_i}(J)=\alpha_i$ for the prime factorization $\sup(J\wedge I)=p_1^{\alpha_1}\cdots p_n^{\alpha_n}$, where $v_p(J)$ is defined to be the minimal value of $\{v_p(c)\mid c\in J\}$. Put $a\equiv\sup(J\wedge I)$, and let $a=\sup A$ be the sup-expression of a by a subset A of G, where G is the group generated by Σ. Then since for any element x of A there exists a finite number of elements c_j such that $x\leq c_1\cup\cdots\cup c_r$, $c_j\in J$, we obtain that $v_{p_i}(x)\geq\mathrm{Min}\{v_{p_i}(c_j)\mid j=1,\cdots,r\}\geq v_{p_i}(J)$, $\alpha_i\geq v_{p_i}(J)$. If we suppose that $\alpha_i>v_{p_i}(J)$ for some i, then we can show that there exists an element u such that $v_{p_i}(u)<\alpha_i$, $u\leq c$, $u\in G\wedge J$. Now we can take an element z with $v_{p_i}(z)=1$, $z\in I_G$. Then we have that $0\leq v_{p_i}(z^{\kappa}u)<\alpha_i$ for a suitable integer κ. Let $z^{\kappa}u=p_i^{\nu}fg^{-1}$, $f\cup g=e$, and f is not divisible by p_i. Then we have $0\leq\nu<\alpha_i$. On the other hand, since $p_i^{\nu}f=z^{\kappa}ug$ is contained in $J\wedge I$, we have $p_i^{\nu}f\leq\sup(J\wedge I)=a$, $\alpha_i=v_{p_i}(a)\leq v_{p_i}(p_i^{\nu}f)=\nu$, a contradiction. Next we prepare the following: Let $p_1,\cdots,p_m$ be a finite number of primes, and let $\kappa_1,\cdots,\kappa_m$ be any fixed integers such that $v_{p_i}(J)\leq\kappa_i$ $(i=1,\cdots,m)$. Then $p_1^{\kappa_1}\cdots p_m^{\kappa_m}\cdot\sup(J\wedge I)\leq\sup J$. This will be proved by induction on m. Then the statement in the first part of this paragraph can be proved as follows: Let J' be the set of the elements c of S such that $v_p(c)\geq v_p(J)$ for every prime p, and let $a\equiv\sup(J\wedge I)=p_1^{\alpha_1}\cdots p_n^{\alpha_n}$ be the prime factorization of a. Then we have $v_{p_i}(c)\geq\alpha_i$ for $i=1,\cdots,n$. Hence $c=p_1^{\lambda_1}\cdots p_n^{\lambda_n}p_{n+1}^{\lambda_{n+1}}\cdots p_m^{\lambda_m}$, where $\lambda_i\geq\alpha_i$ for $i=1,\cdots,n$ and $\lambda_j\geq v_{p_j}(J)$ for $j=n+1,\cdots,m$. Therefore, by using the above argument, we have

$$c=p_1^{\lambda_1-\alpha_1}\cdots p_n^{\lambda_n-\alpha_n}\cdots ap_{n+1}^{\lambda_{n+1}}\cdots p_m^{\lambda_m}\leq\sup(p_1^{\lambda_1-\alpha_1}\cdots p_n^{\lambda_n-\alpha_n}J).$$

Hence $c\leq\sup J$. Hence we have $J'\subseteq J$, $J'=J$, because I is compact. Now since $\{v_p(J)\mid p\in\mathfrak{P}\}$ is almost all non-positive, we can define $\sigma: p$

$\rightarrow p^{\sigma}=v_p(J)$. Then we obtain that $J=J(\sigma)$, as desired.

Now we can prove the followings: Let J and J' be two I-l-ideals of Dedekindian l-semigroup S. Then *in order that there exists an I-l-isomorphism θ from J onto J', it is necessary and sufficient that there exists an element t of S such that $\theta(c)=tc$ for every c in J.* Let σ and σ' be two maps such that $J=J(\sigma)$ and $J'=J(\sigma')$. Then in order that J and J' are I-l-isomorphic, it is necessary and sufficient that $p^{\sigma}=p^{\sigma'}$ for almost all p, and both $\{p^{\sigma} \mid p^{\sigma} \neq p^{\sigma'}\}$ and $\{p^{\sigma'} \mid p^{\sigma'} \neq p^{\sigma}\}$ are finite sets. Now we can prove that $v_p(J(\sigma))=v_p(J(\sigma'))$ for all $p \in \mathfrak{P}$ if and only if $\sigma=\sigma'$. Moreover the set of all I-l-ideals $\ni e$ of Dedekindian l-semigroup forms a Boolean algebra. An integral I-l-ideal J is said to be maximal if there is no I-l-ideal between I and J. Then we can prove that $J_p=\{c \in S \mid v_p(c) \geq 1\}$ is a maximal I-l-ideal for every prime p.

Reference

[1] K. Murata: A characterization of Artinian l-semigroups. Proc. Japan Acad., **47**, 127–131 (1971).