29. *A Characterization of Artinian l-Semigroups*

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The aim of the present note is to generalize Artin's well-known equivalence relation (quasi-equal relation) introduced in commutative rings for some sorts of commutative l-semigroups,[1] and to give a characterization of such l-semigroups by a system of valuations defined on some quotient l-semigroups with compactly generated cones.

1. Let S be a conditionally complete and commutative l-semigroup with unity quantity e, and let I be the cone (integral part) of S. We suppose throughout this paper that I is compactly generated by a compact generator system Σ containing e (cf. [7]), and that S is a quotient semi-group of I by Σ, that is, every element x of Σ is invertible in S and every element c of S can be written as $c=ax^{-1}$, where $a \in I$ and $x \in \Sigma$. If a compactly generated l-semigroup I with a compact generator system Σ is given, we can prove that there exists a quotient l-semigroup of I by Σ, if and only if the following two conditions hold for I and Σ: (i) for any two elements x and y of Σ, there exists an element a of I such that axy is in Σ, and (ii) every element of Σ satisfies the cancellation law. The lattice-structure is naturally introduced in the quotient semigroup, and such a quotient l-semigroup is uniquely determined within isomorphisms over I.

Now it can be proved that the join-semi-lattice generated by Σ is also a compact generator system of I. Hence we may assume, if necessary, that Σ is closed under finite join-operation. If, in particular, S forms a group, we can show that the maximal condition holds for the elements of I. By using this, we can prove the following: in order that a quotient l-semigroup of I by Σ is a group, it is necessary and sufficient that every element of I has a prime factorization and every prime is divisor-free in the sense of the partial-order.

2. In this and the next sections, we let S be a quotient l-semigroup (conditionally complete) of the cone I by a compact generator system Σ of I. The multiplicative group generated by Σ in S will be denoted by G. Then the element of S can be represented as a supremum of a subset of G. For any two elements a, b of S, the set $X_{a,b}$

1) Artin's equivalence relation has been introduced in various l-semigroups by many authors [1], [4], [2], [5], [3], etc. A systematic study was given in [4] and [5].

consisting of the elements x with $bx \leq a$ and $x \in \Sigma$ is not void, and bounded (upper). Hence we can define $a: b \equiv \sup X_{a,b}$, which is called a *residual* of a by b. Then we have the followings: $(\bigcap_{\lambda=1}^{n} a_\lambda): b = \bigcap_{\lambda=1}^{n}(a_\lambda : b)$, $a: (\bigcup_{\lambda=1}^{n} b_\lambda) = \bigcap_{\lambda=1}^{n}(a: b_\lambda)$, $a: (bb') = (a: b): b')$, etc. In particular $a: u = au^{-1}$ for $a \in S$ and $u \in G$. Now it can be shown easily that every element a of S has an upper bound in G. a^* will mean the infimum of the upper bounds in G of a. Then we have $a \leq a^*$, $a^{**} = a^*$, and $a \leq b^*$ implies $a^* \leq b^*$. An element a of S is called closed if $a^* = a$. Then we have that, if a is closed, $a: b$ is closed for every element b of S. Moreover we can show that $a^* = e: (e: a)$, $e: a = e: a^*$, and $a^*b^* \leq (ab)^*$ for every a, b of S. Now we can prove that the set S^* of all closed elements of S forms a quotient l-semigroup of $S^* \wedge I$ by Σ under the multiplication $a^* \circ b^* \equiv (a^*b^*)^* = (ab)^*$. The l-semigroup $(S^*, \circ, \leq)$ is an extension of the *po*-group $(G, \cdot, \leq)$. S^* coincides with the set of all inf A, where A is a non-void subset of G.

We now introduce an equivalence relation $\sim$, called *Artin-equivalence* (quasi-equality) as follows: $a \sim b \Longleftrightarrow a^* = b^*$ ($\Longleftrightarrow e: a = e: b$). Then the set of the classes $S^\wedge$ obtained by the Artin-equivalence forms an l-semigroup naturally, and which is isomorphic to $(S^*, \circ, \leq)$ as l-semigroups.

The cone I of a quotient l-semigroups called *integrally closed* with respect to Σ, if $xu^n \in I$, $x \in \Sigma$, $u \in G$ $(n = 0, 1, 2, \cdots)$ imply $u \in I$. A quotient l-semigroup S is called *Artinian*, if $S^* = (S^*, \circ, \leq)$ forms an l-group. A closed element p of I is called $(\circ)$-*prime*, if whenever $a \circ b \leq p$ implies $a \leq p$ or $b \leq p$ for closed elements a, b in I. It is then easily verified that a closed element p is $(\circ)$-prime if and only if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for $a, b \in I$. Hence $(\circ)$-primes are closed primes. Moreover we can prove that a closed element p is $(\circ)$-prime if and only if $xy \leq p$ implies $x \leq p$ or $y \leq p$ for $x, y \in \Sigma$. Then the following three conditions are equivalent to one another:

(1) *S is Artinian.*

(2) *I is integrally closed with respect to Σ.*

(3) *S^* (or $S^\wedge$) is isomorphic to the l-group $[\boldsymbol{Z}, \mathfrak{P}]$ consisting of all $(\alpha_p \mid p \in \mathfrak{P})$ such that $\alpha_p \in \boldsymbol{Z}$ (the ring of the integers) and $\alpha_p = 0$ for almost all $p \in \mathfrak{P}$ (the set of the $(\circ)$-primes).*

If S is an Artinian l-semigroup, every $(\circ)$-prime is prime, and every prime contains a $(\circ)$-prime. Moreover we have that an element p is $(\circ)$-prime if and only if p contains no prime except p itself.

3. In this section we assume that Σ is closed under finite join-operation and multiplication. If the product of any two elements of Σ can be written as a join of a finite number of elements of Σ ([6]), then evidently Σ is closed under multiplication. In this case we can

show that every element of G is compact, and G is a compact generator system of S.

A map $x \to v(x)$ from $G=\{x, y, z, \cdots\}$ into $\boldsymbol{Z}$ is called here a *valuation* of S, if it satisfies the following three conditions:

(a) $x \leq y$ implies $v(x) \geq v(y)$.

(b) $v(xy)=v(x)+v(y)$.

(c) $v(x \cup y)=\mathrm{Min}\{v(x), v(y)\}$.

If S is an Artinian l-semigroup, every element of S^* has the $(\circ)$-prime factorization, and in particular so is the element of $G: x=\mathring{\prod} p^{n_p}$, $n_p \in \boldsymbol{Z}$. Then the map $x \to v_p(x) \equiv n_p$ from G into $\boldsymbol{Z}$ satisfies the above three conditions. For every element a of S (not necessarily Artinian), we let U_a be the set of the elements such that $x \leq a$ and $x \in G$. $v(a)$ will mean $\mathrm{Min}\{v(x) \mid x \in U_a\}$, and which is called a valuation of a. Then we can show that for arbitrary sup-expression $a=\sup A, A \subseteq G$, of any fixed element $a \in S$, there exists an element z in A such that $v(a)=v(z)$. By using this, we can show that the conditions (a), (b), and (c) hold for the elements of S. Let $\mathfrak{P}$ be the set of the $(\circ)$-prime elements in I, and let $I(v_p)$ be the elements a such that $a \in S$ and $v_p(a)=\mathrm{Min}\{v_p(x) \mid x \in U_a\} \geq 0$. Then we can prove that if S is Artinian, then $I=\wedge_{p \in \mathfrak{P}} I(v_p)$.[2)] Let S be Artinian. Then the set of $(\circ)$-primes p with $v_p(a) \neq 0$ is finite for an arbitrary fixed element a of S. Moreover, if $p_1 \neq p_2$ in $\mathfrak{P}$, there exists an element x such that $x \in \Sigma$, $v_{p_1}(x)>0$ and $v_{p_2}(x)=0$. In fact, such an element x can be taken as $x \leq p_1$ and $x \nleq p_1 \circ p_2$.

4. Our purpose of this section is to characterize the Artinian l-semigroup by the properties mentioned in the last part of Section 3. The results obtained in this section are analogous to those of [8; Chap. 4].

Let L be an l-semigroup with an identity e. We now suppose that L is conditionally complete and compactly generated by $G=\{x \in L \mid xx' = e$ for some $x' \in L\}$, and G is closed under finite join-operation. A map v from L into $\boldsymbol{Z}$ is called a valuation of L, if it satisfies the three properties (a), (b), and (c) in Section 3. We now assume that there exists a family $\mathfrak{B}=\{v\}$ of valuations which satisfies the following three conditions:

(A) L is a quotient l-semigroup of $I=\wedge_{v \in \mathfrak{B}} I(v)$ by $I_G=I \wedge G$, where $I(v)$ is the set of the elements a of L such that $v(a) \geq 0$.

(B) The set consisting of v in $\mathfrak{B}$ with $v(a) \neq 0$ is finite for each element a of L.

(C) If $v_1 \neq v_2$ in $\mathfrak{B}$, there exists an element x such that $x \in I_G$, $v_1(x)>0$ and $v_2(x)=0$.

Let $v_0, v_1, \cdots, v_n$ be any finite number of valuations of $\mathfrak{B}$. Then

2) $\wedge$ means intersection.

we can show that (1) there exists an element $x \in I_G$ such that $v_0(x)=0$ and $v_i(x)>0$ for $i=1, \cdots, n$, and (2) there exists $y \in I_G$ such that $v_0(y)>0$ and $v_i(y)=0$ for $i=1, \cdots, n$. Moreover we can show that for a finite number of valuations $v_1, \cdots, v_m$ and for any fixed element a of L, there exists an element u of G such that $u \leq a$, and $v_i(a)=v_i(u)$ for $i=1, \cdots, m$. By using this we obtain that $v((e: a): a)=0$ for every element $a \in L$ and every valuation $v \in \mathfrak{V}$, where the residuation in L is defined similarly as in Section 2. An element c of L is said to be *low* if $v(c)=0$ for all $v \in \mathfrak{V}$. Then we have that (1) c is low if and only if $e: c=e$, (2) if c is low, then $e: ac=e: a$ for every $a \in L$, and (3) if every element of I is compact, e is the only low element of L. In L Artin-equivalence is defined in the obvious way. Then we can prove that a and b are Artin-equivalent if and only if $v(a)=v(b)$ for all $v \in \mathfrak{V}$. $\mathfrak{V}$ is said to be normal, if there exists an element u of G such that $v(u)=1$ for each valuation $v \in \mathfrak{V}$. Now let $\mathfrak{V}$ be normal, let $v, v_1, \cdots, v_n$ be a finite number of valuations in $\mathfrak{V}$ such that $v \neq v_i$ for $i=1, \cdots, n$, and let $v(u_0)=1, u_0 \in G$. Next we let $v_{n+1}, \cdots, v_m$ be the set of all valuations such that $v(u_0) \neq 0$, $v_j \neq v_1, \cdots, v_n$ for $j=n+1, \cdots, m$. Since we can choose an element u of I_G such that $v(u)=0$, $v_1(u)>0, \cdots, v_n(u)>0$, $v_{n+1}(u)>0, \cdots, v_m(u)>0$, we obtain $v_i(u^\rho u_0)>0$ for a sufficiently large integer ρ $(i=1, \cdots, m)$. Then it can be shown that $u^\rho u_0 \leq e$. Hence, by taking elements x_k of I_G such that $v_k(x_k)=0$, $v(x_k)>1$, and $v_j(x_k)>0$ $(j \neq k, 1 \leq k \leq n)$, we can prove that the element $t=\bigcup_{k=1}^n x_k \cup u^\rho u_0$ is in I_G and satisfies $v(t)=1$, $v_i(t)=0$ for $i=1, \cdots, n$. Moreover, for any fixed $v \in \mathfrak{V}$, we can show the existence of the element $s(v) \in I_G$ such that $v(s(v))=1$ and $v'(s(v))=0$ for all v' with $v' \neq v$, $v' \in \mathfrak{V}$. By using the above facts, we can prove that *L is an Artinian l-semigroup, that is, the set $L^\wedge$ of all classes obtained by the Artin-equivalence relation forms an l-group, which is isomorphic to $(\boldsymbol{Z}, \mathfrak{V})$ as l-groups, where $(\boldsymbol{Z}, \mathfrak{V})$ is the l-group consisting of all $(\alpha_v \mid v \in \mathfrak{V})$ such that $\alpha_v \in \boldsymbol{Z}$ and $\alpha_v=0$ for almost all $v \in \mathfrak{V}$. Hence L is Artinian if and only if it has a system of valuations with the properties* (A), (B) *and* (C). Now it can be shown that $p_v=\sup\{u \in I(v) \wedge G \mid v(u)>0\}$ is a prime element in $I(v)$, and $p(v)=p_v \cap e$ is a prime element in I. Since $p(v_1) \neq p(v_2)$ for $v_1 \neq v_2$, and $p(v) \sim s(v)$ (Artin-equivalence), we obtain that every class in $L^\wedge$ is factored into a product of a finite number of $K(p(v_i))$, the class containing $p(v_i)$, and the factorization is unique apart from its commutativity. In other words, $L^\wedge$ is the (restricted) direct product $\Pi \bigotimes_{v \in \mathfrak{V}} K(p(v))$.

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