71. A Note on the Number of Generators of an Ideal

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Through this note, we mean by a ring a commutative ring with identity 1. Let R be a noetherian ring and A be an ideal of R. O. Forster showed that, if AR_M is generated by at most r elements for any maximal ideal M of R, then A is generated by at most $r+\text{Alt.}\,R$ elements, where Alt. R is the Krull dimension of R (cf. O. Forster [1]). In this note, we shall study the number of generators of an ideal and improve the above Forster's result, that is:

Theorem 1. *Let R be a ring and A be a finitely generated ideal of R. Assume that:* (1) *there are only a finite number of maximal ideals of R which contain A and* (2) *AR_M is generated by at most r elements for any maximal ideal M of R. Then A is generated by at most $r+1$ elements.*

Theorem 2. *Let R be a noetherian ring and A be an ideal of R such that* $\text{Alt.}\,R/A<\infty$. *Assume AR_M is generated by at most r elements for any maximal ideal M of R. Then A is generated by at most* $r+\text{Alt.}\,R/A+1$ *elements.*

To prove these theorems we need the following lemmas.

Lemma 1. *Let R be a ring. Assume $0=Q_1\cap\cdots\cap Q_n$ be an irredundant decomposition of zero ideal of R (not necessarily primary decomposition). If $Q_1+Q_j=R$ $(j=2,\cdots,n)$, then Q_1 is a principal ideal.*

Proof. Since $Q_1\oplus Q_2Q_3\cdots Q_n=R$, we can take $x\in Q_1$ and $y\in Q_2\cdots Q_n$ such that $x+y=1$. For any element $z\in Q_1$, $z=zx+zy=zx$, so we have $Q_1=xR$.

Lemma 2. *Let R be a ring and A be a finitely generated ideal which contains an ideal B. If $AR_M=BR_M$ for any maximal ideal M which contains A, then $A=B$ or $A=xR+B$ for some element x of A.*

Proof. Since A is finitely generated, $AR_M=BR_M$ implies $B:A\not\subset M$ for any maximal ideal M which contains A. So we have $(A\cap(B:A))R_M=BR_M$ for any maximal ideal M of R, hence $B=A\cap(B:A)$. If $B:A=R$ then $B=A$. If $B:A\neq R$ then $A+(B:A)=R$ since $B:A\not\subset M$ for any maximal ideal M which contains A. So Lemma 1 implies $A=B+xR$ for some $x\in A$ by considering R/B and A/B.

Lemma 3. *Let R be a ring and A be an ideal of R. Assume that:* (1) *there are only a finite number of maximal ideals $M_1,\cdots,M_n$ which contain A and* (2) *AR_{M_i} is generated by at most r elements for every i.*

Then there are elements $x_1, \cdots, x_r$ *of* A *such that* $(x_1, \cdots, x_r)R_{M_i} = AR_{M_i}$ $(i=1, \cdots, n)$.

Proof. Choose x_{ij} of A such that $(x_{i1}, \cdots, x_{ir})R_{M_i} = AR_{M_i} (i=1, \cdots, n)$ and take elements $\alpha_1, \cdots, \alpha_n$ of R such that $\alpha_i \notin M_i$ and $\alpha_i \in \bigcap_{j \neq i} M_j (i=1, \cdots, n; j=1, \cdots, n)$. Put $x_j = \sum_{i=1}^{n} x_{ij} \alpha_i$ then $x_j = x_{ij} \alpha_i$ $(\text{mod } AM_i)(i=1, \cdots, n)$. So we have $(x_1, \cdots, x_r)R_{M_i} = (x_{i1}, \cdots, x_{ir})R_{M_i} = AR_{M_i}(i=1, \cdots, n)$.

Remark 1. By Lemma 3, when $(R, M_1, \cdots, M_n)$ is a quasi-semi-local ring and if AR_{M_i} is generated by at most r elements for every i, A is generated by r elements. So if R_{M_i} is noetherian for every i, then R is noetherian. This is (E1. 2) of Appendix in Nagata [3].

Proof of Theorem 1. Obvious by Lemma 3 and Lemma 2.

Corollary 1. *Let* R *be a ring* (*not necessarily noetherian*) *and* M *be a maximal ideal which is finitely generated. If* MR_M *is generated by* r *elements, then* M *is generated by at most* $r+1$ *elements.*

Corollary 2. *Let* R *be a noetherian ring and* A *be an ideal of* R *such that* Alt. $R/A=0$. *If* AR_M *is generated by at most* r *elements for any maximal ideal* M *of* R *then* M *is generated by at most* $r+1$ *elements.*

From this corollary, we have the well known

Corollary 3. *Let* R *be a Dedekind ring then any ideal of* R *is generated by at most two elements.*

Let R be a noetherian ring and A be an ideal of R such that Alt. $R/A < \infty$. We use the following notation:

$$\text{Spec } R = \{\text{the set of all prime ideals of } R\},$$
$$V(A) = \{P \in \text{Spec } R \mid P \supset A\},$$
$$B(t) = \sum_{a_i \in A} ((a_1, \cdots, a_{t-1}) : A).$$

Lemma 4. *Let* P *be a prime ideal. Then* $\mu(AR_P) \geqq t$ *if and only if* $P \supset B(t)$, *where* $\mu(AR_P)$ *is the minimal number of generators of* AR_P.

Proof. $P \supset B(t)$ implies $P \supset (a_1, \cdots, a_{t-1}) : A$ for any $a_1, \cdots, a_{t-1}$ of A. This means that $AR_P \neq \sum_{i=1}^{t-1} a_i R_P$ for any elements $a_1, \cdots, a_{t-1}$ of A, thus we have $\mu(AR_P) \geqq t$. Conversely $\mu(AR_P) \geqq t$ means $AR_P \supsetneqq \sum_{i=1}^{t-1} a_i R_P$ for any elements $a_1, \cdots, a_{t-1}$ of A so we have $\sum_{i=1}^{t-1} a_i R_p : AR_P \subsetneqq PR_P$ thus $P \supset \sum_{i=1}^{t-1} a_i R : A$ for any $a_1, \cdots, a_{t-1}$ of A, so $P \supset B(t)$. This completes the proof.

Let R and A be as above. For any $P \in \text{Spec } R$, put

$$f_P(A) = \begin{cases} (AR_P) + \text{Alt. } R/P & \text{if } AR_P \neq 0 \\ 0 & \text{if } AR_P = 0, \end{cases}$$

$$f(A) = \sup_{P \in V(A)} f_P(A) \quad \text{and} \quad g(A) = \sup_{P \in \text{Spec } R} f_P(A).$$

Lemma 5. *Assume that* R *is noetherian and that* Alt. $R/A < \infty$. *Put* $S = \{P \in V(A) \mid f_P(A) = f(A)\}$ *and* $T = \{P \in \text{Spec } R \mid f_P(A) = g(A)\}$. *Then* (1) S *is a finite set if* $f(A) > 0$, (2) T *is a finite set if* $g(A) > 0$ *and* Alt. $R < \infty$.

Proof. $f(A)$ is finite since Alt. R/A is finite and $g(A)$ is finite since Alt. R is finite.

(1) For any $P \in S$, put $t=\mu(AR_P)$ then $P \supset A+B(t)$ by Lemma 4, so there exists a minimal prime ideal P' of $A+B(t)$ such that $P \supset P'$. $t=\mu(AR_P) \geqq \mu(AR_{P'}) \geqq t$ implies $\mu(AR_P)=\mu(AR_{P'})=t$. On the other hand, $\mu(AR_P)+\text{Alt. } R/P \geqq \mu(AR_{P'})+\text{Alt. } R/P'$ since $P \in S$, so we have Alt. $R/P=$ Alt. R/P', hence $P=P'$. Thus S is a finite set.

(2) For any $P \in T$, set $t=\mu(AR_P)$ then P must be a minimal prime ideal of $B(t)$ in the same way as in (1), so T is finite.

Lemma 6 (Forster's lemma). *Let $P_1, \cdots, P_n$ be prime ideals and A an ideal. If $AR_{P_i} \neq 0 (i=1, \cdots, n)$ then there exists an element x of A such that $xR_{P_i} \not\subset AP_iR_{P_i} (i=1,2, \cdots n)$.*

Proof. We may assume $P_i \not\subset P_j (j>i)$. We prove this lemma by induction on n. If $n=1$, it is obvious. If $n>1$, we can take an element y of A such that $yR_{P_i} \not\subset AP_iR_{P_i}$ $(i=1, \cdots, n-1)$ by the hypothesis of induction. If $yR_{P_n} \not\subset AP_nR_{P_n}$, put $x=y$. If $yR_{P_n} \subset AP_nR_{P_n}$, take elements $a, z_1, \cdots, z_{n-1}$ of R such that $a \in A, aR_{P_n} \not\subset AP_nR_{P_n}$ and $z_i \in P_i - P_n$ $(i=1, \cdots, n-1)$. Put $z=az_1 \cdots z_{n-1}$ and $x=y+z$, then $xR_{P_i} \not\subset AP_iR_{P_i}$ $(i=1, \cdots, n)$.

Proof of Theorem 2. Let notations be as in Lemma 5. We shall show that A is generated by at most $f(A)+1$ elements, by induction on $f(A)$. If $f(A)=0$, then $AR_P=0$ for any $P \in V(A)$ so $AR_M=0$ for any maximal ideal which contains A. Thus A is principal by Lemma 2. If $f(A)>0$, then we can take an element x of A such that $\mu(AR_P/xR_P) =\mu(AR_P)-1$ for any $P \in S$ by Lemma 5 and Lemma 6. Let $\bar{R}=R/xR$, $\bar{P}=P/xR (P \in V(A))$ and $\bar{A}=A/xR$. If $P \in S$ then $\mu(AR_P/xR_P) + \text{Alt. } (\bar{R}/\bar{P}) = \mu(AR_P)-1+\text{Alt. } (R/P)=f(A)-1$. If $P \notin S$ then $\mu(AR_P/xR_P)+\text{Alt. } (\bar{R}/\bar{P}) \leqq \mu(AR_P)+\text{Alt. } (R/P)<f(A)$ so we have $f(\bar{A}) =f(A)-1$ since $\bar{A}\bar{R}_{\bar{P}}=AR_P/xR_P$. Thus $\bar{A}$ is generated by at most $f(\bar{A})+1=f(A)$ elements, so A is generated by at most $f(A)+1$ elements. For any $P \in V(A)$ and for any maximal ideal M of R, we have Alt. $(R/P) \leqq$ Alt. (R/A) and Sup $\mu(AR_P) \leqq$ Sup $\mu(AR_M)$, so the proof is complete.

The following proposition is an improvement of Corollary 1 and Satz 4 of [1], in a special case.

Proposition 1. *Let R be a noetherian ring, M be a maximal ideal of R and Q be an M-primary ideal. Assume $\mu(QR_M)>$ Alt. R. Then Q is generated by $\mu(QR_M)$ elements.*

Proof. Put $\mu(QR_M)=r$ and $\sqrt{0}=\bigcap_{i=1}^{n_0} P(0,i)$ where $P(0,i)$ is a minimal prime of zero. We may assume $P(0,1) \subset M$. By virtue of Proposition 2 of Chap. 2 of [2], we can take elements $x_1, \cdots, x_r$ of A satisfying the following conditions:

(1) $\sqrt{(x_1, \cdots, x_i)}=\bigcap_{j=1}^{n_i} P(i,j)$ and $P(i,1) \subset M (0 \leqq i \leqq r)$ where each

$P(i,j)$ is a minimal prime ideal of $(x_1,\cdots,x_i)$,

(2) $x_{i+1}\notin(x_1,\cdots,x_i)+QM,(0\leqq i\leqq r)$,

(3) $x_{i+1}\notin P(i,j),(j=2,\cdots,n_i)$ if $M=P(i,1),(0\leqq i\leqq r)$,

(3′) $x_{i+1}\notin P(i,j),(j=1,\cdots,n_i)$ if $M\neq P(i,1),(0\leqq i\leqq r)$.

We can take these elements $x_1,\cdots,x_r$ of Q. For:

$Q\subset(x_1,\cdots,x_s)+QM$ implies $QR_M=(x_1,\cdots,x_s)R_M+QMR_M$, hence $QR_M=(x_1,\cdots,x_s)R_M$.

(2) implies $(x_1,\cdots,x_r)R_M=QR_M$ since $\mu(QR_M)=r$, so we have $P(r,1)=M$. If there exists $P(r,j),(j\geqq 2)$ then we have a chain of prime ideals

$$P(r,j)\supsetneqq P(r-1,j_{r-1})\supsetneqq\cdots\supsetneqq P(1,j_1)\supsetneqq P(0,j_0)$$

by (3) and (3′). So height $P(r,j)\geqq r$ for any $j,(j\geqq 2)$. This contradicts Alt. $R<r$. Hence $\sqrt{(x_1,\cdots,x_r)}=P(r,1)=M$. So we have $Q=(x_1,\cdots,x_r)$ since $QR_M=(x_1,\cdots,x_r)R_M$ and $(x_1,\cdots,x_r)$ is an M-primary ideal.

Remark 2. If R is noetherian with Alt. $R<\infty$ and AR_M is generated by at most r elements for any maximal ideal M, then we may prove that A is generated by at most $g(A)$ elements by using induction on $g(A)$. (cf. Forster, [1]). When Alt. $(R/A)=$Alt. R, Forster's result (Satz 2 of [1]) is better than our Theorem 2.

References

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