70. *On the Minimal Group Congruence on the Tensor Product of Archimedean Commutative Semigroups*

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(Comm. by Kenjiro SHODA, M. J. A., March 12, 1971)

By the tensor product $X \otimes Y$ of commutative semigroups X and Y we mean the quotient semigroup $F(X\times Y)/\delta$ where $F(X\times Y)$ is the free commutative semigroup on the set $X\times Y$ and δ is the smallest congruence relation for which:

$$(x_1+x_2, y)\delta(x_1, y)+(x_2, y)$$

and

$$(x, y_1+y_2)\delta(x, y_1)+(x, y_2)$$

hold for all $x_1, x_2, x \in X$ and $y, y_1, y_2 \in Y$.

If α and β are congruences on semigroups X and Y, then $\alpha\otimes\beta$, which is called the tensor product of congruences α and β, is the smallest congruence on the tensor product $X\otimes Y$ containing all pairs $(x_1\otimes y_1, x_2\otimes y_2)$ such that

$$(x_1, x_2)\in\alpha \quad \text{and} \quad (y_1, y_2)\in\beta, \qquad \text{(see, [2])}.$$

A congruence δ on a semigroup X is called a group congruence if X/δ is a group. W. D. Munn [4] proved that a relation α defined on an inverse semigroup X by the rule that $x_1 \,\alpha\, x_2$ $(x_1, x_2 \in X)$ if and only if $x_1+e=x_2+e$ for some idempotent e of X is the minimal group congruence on X. The author [3] proved that X and Y are commutative inverse semigroups which possess the minimal group congruences α and β, respectively, then the tensor product $X\otimes Y$ possesses the minimal group congruence and it is the tensor product $\alpha\otimes\beta$. In this note we shall give such a property in the case when X and Y are archimedean commutative semigroups with idempotents, where a commutative semigroup X is called archimedean if for every $a, b\in X$, there exist elements $x, y\in X$ and positive integers m, n such that

$$ma=b+x \quad \text{and} \quad nb=a+y, \qquad \text{(see, [5] or [1])}.$$

Lemma 1 ([5] Theorem 3). *An archimedean commutative semigroup can contain at most one idempotent.*

Lemma 2. *Let X be an archimedean commutative semigroup with an idempotent e and let a relation α be defined on X by the rule that $x_1 \,\alpha\, x_2$ $(x_1, x_2\in X)$ if and only if*

$$x_1+e=x_2+e.$$

Then α is a congruence and X/α is a group. Further, if γ is any con-

gruence on X with the property that X/γ is a group, then $\alpha\subseteq\gamma$, that is, α is the minimal group congruence on X.

Proof. It is clear that α is a congruence on X. We prove that the quotient semigroup X/α is a group. For any $x\in X$, let $x\alpha$ denote the α-class of X containing x; thus $x\to x\alpha$ is the natural homomorphism of X onto X/α. Since for any $x\in X$,

$$x+e=x+e+e,$$

we have

$$x\alpha=(x+e)\alpha,$$

and so

$$x\alpha+e\alpha=(x+e)\alpha=x\alpha.$$

Since X is archimedean, for any $x\in X$, there exist element $z\in X$ and positive integer m such that

$$x+z=me=e.$$

Then we have

$$x\alpha+z\alpha=(x+z)\alpha=e\alpha.$$

Therefore X/α is a group.

Let γ be a congruence on X with the property that X/γ is a group. We shall prove that if two elements of X lie in the same α-class they must lie in the same γ-class. Let the γ-class of X containing x be denoted by $x\gamma$. Suppose that $x_1\alpha x_2 (x_1, x_2\in X)$. Then we have

$$x_1+e=x_2+e,$$

and so

$$x_1\gamma+e\gamma=(x_1+e)\gamma=(x_2+e)\gamma=x_2\gamma+e\gamma.$$

Since $e\gamma$ is an idempotent of the group X/γ, it is the identity. Hence we have $x_1\gamma=x_2\gamma$, which shows that $\alpha\subseteq\gamma$. This completes the proof of Lemma 2.

The following lemma is easily seen.

Lemma 3. *If X and Y are archimedean commutative semigroups, each having idempotent e and f, then the tensor product $X\otimes Y$ is an archimedean commutative semigroup with unique idempotent $e\otimes f$.*

Lemma 4 ([2] Corollary 3.5)**.** *Let γ and δ be congruences on semigroups X and Y, respectively. Then $X/\gamma\otimes Y/\delta$ is isomorphic to $(X\otimes Y)/(\gamma\otimes\delta)$.*

Theorem 5. *If X and Y are archimedean commutative semigroups, each having idempotent, which possess the minimal group congruences α and β, respectively, then the tensor product $X\otimes Y$ of X and Y possesses the minimal group congruence and it is the tensor product $\alpha\otimes\beta$ of α and β.*

Proof. Let e and f be idempotents of archimedean commutative semigroups X and Y, respectively. Then by Lemma 3, the tensor product $X\otimes Y$ is an archimedean semigroup with unique idempotent

$e\otimes f$. By Lemma 2, we have

$$x_1\alpha x_2\,(x_1, x_2\in X) \quad \text{if and only if } x_1+e=x_2+e$$

and

$$y_1\beta y_2\,(y_1, y_2\in Y) \quad \text{if and only if } y_1+f=y_2+f.$$

Since α and β are group congruences on X and Y, respectively, it follows from Lemma 4 that the tensor product $\alpha\otimes\beta$ is a group congruence on the tensor product $X\otimes Y$. If δ is the minimal group congruence on the tensor product $X\otimes Y$, then

$$\delta\subseteq\alpha\otimes\beta.$$

To show the converse inclusion, let x_1 and x_2 be any elements of X such that $(x_1, x_2)\in\alpha$. Then by Lemma 2, we have

$$x_1+e=x_2+e.$$

Then for any element $y\in Y$, we have

$$x_1\otimes y+e\otimes y=(x_1+e)\otimes y=(x_2+e)\otimes y=x_2\otimes y+e\otimes y.$$

Since $e\otimes y$ is idempotent of the tensor product $X\otimes Y$, it follows from Lemma 3 that

$$e\otimes y=e\otimes f.$$

Hence we have

$$x_1\otimes y+e\otimes f=x_2\otimes y+e\otimes f$$

for any element $y\in Y$. This means that

$$(x_1\otimes y, x_2\otimes y)\in\delta$$

for any element $y\in Y$.

Similarly, $(y_1, y_2)\in\beta$ implies

$$(x\otimes y_1, x\otimes y_2)\in\delta$$

for any element $x\in X$.

Therefore $(x_1, x_2)\in\alpha$ and $(y_1, y_2)\in\beta$ imply

$$(x_1\otimes y_1, x_2\otimes y_1)\in\delta$$

and

$$(x_2\otimes y_1, x_2\otimes y_2)\in\delta$$

and so

$$(x_1\otimes y_1, x_2\otimes y_2)\in\delta.$$

Thus we have

$$\alpha\otimes\beta\subseteq\delta.$$

This completes the proof of Theorem 5.

Remark. T. Head, in his recent paper "Commutative semigroups having greatest regular images" (to appear), has given the following property:

An archimedean commutative semigroup has a greatest group image if and only if it contains an idempotent.

Thus it follows from this and Lemma 3 that the first half of Theorem 5 holds.

References

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