

65. On the Numerical Range of an Operator

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1. Introduction. In this paper, an *operator* T means a bounded linear operator acting on a complex Hilbert space H .

Following after Halmos [6] we define the *numerical range* $W(T)$ and the *numerical radius* $w(T)$ of T as follows:

$$W(T) = \{(Tx, x) ; \|x\| = 1\}$$

and

$$w(T) = \sup \{|\lambda| ; \lambda \in W(T)\}.$$

$W(T)$ is convex and the closure $\overline{W(T)}$ of $W(T)$ contains the *spectrum* $\sigma(T)$ of T ; $w(T)$ is a norm equivalent to the operator norm $\|T\|$ which satisfies

$$\frac{1}{2} \|T\| \leq w(T) \leq \|T\|$$

and the *power inequality* ([3]):

$$w(T^n) \leq w(T)^n \quad (n=1, 2, \dots).$$

Definition 1 ([6]). An operator T is said to be *convexoid* if

$$\overline{W(T)} = \text{co } \sigma(T),$$

where $\text{co } \sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of T .

Definition 2 ([6]). An operator T is said to be *spectraloid* if

$$w(T) = r(T),$$

where $r(T)$ means the *spectral radius* of T :

$$r(T) = \sup \{|\lambda| ; \lambda \in \sigma(T)\}.$$

By [4], it is known that T is a spectraloid if and only if

$$w(T)^n = w(T^n) \quad (n=1, 2, \dots).$$

Definition 3 ([6]). An operator T is said to be *normaloid* if

$$\|T\| = r(T),$$

or equivalently

$$\|T\|^n = \|T^n\| \quad (n=1, 2, \dots).$$

The classes of normaloids and convexoids are both contained in the class of spectraloids (cf. [6; p. 115]).

Definition 4 ([1]). A unitary operator U is said to be *cramped* if $\sigma(U)$ is contained in some semicircle:

$$\sigma(U) \subset \{e^{i\theta} ; \theta_1 \leq \theta \leq \theta_2, \theta_2 - \theta_1 < \pi\}.$$

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators acting on

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H. The set of all regular elements of $\mathbf{B}(H)$ will be denoted by $\mathbf{G}(H)$.

$$\Sigma = \{f \in \mathbf{B}(H)^*; f(1) = 1 = \|f\|\}$$

will be called the *state space* of $\mathbf{B}(H)$.

Our main results of this paper are as follows:

(i) Convexoids operators are characterized: T is convexoid if and only if $T - \lambda$ is a spectraloid for every complex λ (Theorem 3).

(ii) An elementary proof of Rota's theorem (Theorem 4) basing on the idea of Hildebrandt.

(iii) An another simple proof of Hildebrandt's theorem (Theorem 5).

(iv) A comment on a theorem of Berberian quoted in a recent paper of Istrăţescu (Theorem 8).

We should express here our cordial thanks to Professors H. Choda and M. Nakamura who encouraged us to prepare this paper.

2. Very recently, J. P. Williams [14; Theorem 1] proves an interesting theorem:

Theorem 1 (Williams). $0 \in \overline{W(T)}$ if and only if $|\lambda| \leq \|T - \lambda\|$ for all complex λ .

In this section, we shall give a variant of Williams' theorem which replaces the norm by the numerical radius:

Theorem 2. $0 \in \overline{W(T)}$ if and only if $|\lambda| \leq w(T - \lambda)$ for all complex λ .

Our proof is an imitation of that of Williams. If $0 \in \overline{W(T)}$, then there exists a state $f \in \Sigma$ such as $f(T) = 0$ by [2], [11] or [14]; hence we have for every complex

$$|\lambda| = |f(T - \lambda)| \leq \sup_{g \in \Sigma} |g(T - \lambda)| = w(T - \lambda),$$

which proves the necessity.

Conversely, if $|\lambda| \leq w(T - \lambda)$ for every complex λ , then we have

$$|\lambda| \leq w(T - \lambda) \leq \|T - \lambda\|,$$

which satisfies the requirement of Williams' theorem; hence by the theorem of Williams we have $0 \in \overline{W(T)}$.

Williams [14; Corollary 1 of Theorem 1] and Hildebrandt [7; Satz 5] pointed out

$$(1) \quad \overline{W(T)} = \bigcap_{\mu} \{\lambda; |\lambda - \mu| \leq \|T - \mu\|\}.$$

Via similar reasoning for Theorem 2, we have

$$(2) \quad \overline{W(T)} = \bigcap_{\mu} \{\lambda; |\lambda - \mu| \leq w(T - \mu)\}.$$

[14; Corollary 2 of Theorem 1] is incorrect, since a convexoid is not always a normaloid by an example of Halmos [6; Problem 174]. The following theorem gives a characterization of convexoids based on the idea of Williams:

Theorem 3. An operator T is convexoid if and only if $T - \lambda$ is a spectraloid for every complex λ .

Proof. At first, we shall notice

$$(3) \quad \text{co } \sigma(T) = \bigcap_{\mu} \{ \lambda; |\lambda - \mu| \leq r(T - \mu) \}.$$

If $T - \lambda$ is a spectraloid for every complex λ , $w(T - \lambda) = r(T - \lambda)$ by Definition 2. Hence we have by (2) and (3)

$$\begin{aligned} \text{co } \sigma(T) &= \bigcap_{\mu} \{ \lambda; |\lambda - \mu| \leq r(T - \mu) \} \\ &= \bigcap_{\mu} \{ \lambda; |\lambda - \mu| \leq w(T - \mu) \} = \overline{W(T)}, \end{aligned}$$

which states that T is a convexoid.

Conversely, if T is a convexoid, then $w(T - \lambda) = r(T - \lambda)$ for every λ since we have

$$\begin{aligned} \overline{W(T - \lambda)} &= \overline{W(T)} - \lambda \\ &= \text{co } \sigma(T) - \lambda = \text{co } \sigma(T - \lambda); \end{aligned}$$

hence $T - \lambda$ is a spectraloid for every complex λ .

“If” part of Theorem 3 is a generalization of a theorem of [7*] and [10]: T is a convexoid if $T - \lambda$ is normaloid for every λ .

At this end, we wish to remark that Theorems 2 and 3 are valid for Banach algebras as much as Williams [14].

3. In the theory of spectra of general operators, the following two theorems due to G. C. Rota [9] and S. Hildebrandt [7] have fundamental importance:

Theorem 4 (Rota). *For every operator T , we have*

$$(4) \quad r(T) = \inf_{S \in G(H)} \|S^{-1}TS\|.$$

Theorem 5 (Hildebrandt). *For every operator T , we have*

$$(5) \quad \text{co } \sigma(T) = \bigcap_{S \in G(H)} \overline{W(S^{-1}TS)}.$$

Additionally, we shall point out that Rota’s theorem implies

$$(6) \quad r(T) = \inf_{S \in G(H)} w(S^{-1}TS),$$

since we have

$$r(T) = r(S^{-1}TS) \leq w(S^{-1}TS) \leq \|S^{-1}TS\|$$

for every invertible S .

In this section, we shall give an elementary proof of Rota’s theorem based on the idea of Hildebrandt [7]. Our proof is taken from a seminar talk of Prof. H. Choda to whom the authors are indebted. Since Williams [13] points out that Rota’s theorem implies Hildebrandt’s theorem, our proof shows that the theorems of Rota and Hildebrandt are nearly equivalent.

We shall begin with the following lemma due to Hildebrandt:

Lemma [7; Lemma 1]. *If*

$$(7) \quad 0 \leq \|T^n\| \leq \theta^n$$

for all sufficiently large n , then for every $\delta > \theta$ we can find an invertible selfadjoint operator S such that

$$(8) \quad \|STS^{-1}\| \leq \delta.$$

For the sake of completeness, we shall give here the proof of the lemma due to Hildebrandt with a minor modification. Define

$$R = \sum_{n=0}^{\infty} \frac{1}{\delta^{2n}} T^{*n} T^n.$$

Then we have clearly $R \in \mathcal{B}(H)$ by (7) and $R \geq 1$. Let S be the positive square root of R . Then S is invertible and hermitean, and we have

$$T^* R T = \delta^2 (R - I) \leq \delta^2 R.$$

Hence we have

$$\begin{aligned} \|STS^{-1}\|^2 &= \|S^{-1}T^*SSTS^{-1}\| \\ &= \|S^{-1}T^*RTS^{-1}\| \\ &\leq \delta^2 \|S^{-1}RS^{-1}\| = \delta^2, \end{aligned}$$

as desired.

Now, we shall give a proof of Rota's theorem. Since

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n},$$

for every $\varepsilon > 0$ and all sufficiently large n , we have

$$\|T^n\| \leq (r(T) + \varepsilon)^n.$$

Using the lemma, we have

$$\|STS^{-1}\| \leq r(T) + 2\varepsilon.$$

Since ε is arbitrary, we have (4).

Now, we shall give a simple proof of Hildebrandt's theorem. Our proof is based on (1) and (3):

$$\begin{aligned} \text{co } \sigma(T) &= \bigcap_{\mu} \{ \lambda; |\lambda - \mu| \leq r(T - \mu) \} \\ &= \bigcap_{\mu} \{ \lambda; |\lambda - \mu| \leq \inf_{S \in \mathcal{G}(H)} \|S^{-1}TS - \mu\| \} \\ &= \bigcap_{\mu} \bigcap_{S \in \mathcal{G}(H)} \{ \lambda; |\lambda - \mu| \leq \|S^{-1}TS - \mu\| \} \\ &= \bigcap_{S \in \mathcal{G}(H)} \overline{W(S^{-1}TS)}. \end{aligned}$$

Remark. (1) Hildebrandt himself gives a simple proof of Theorem 5 after Lemma, cf. [7; Satz 4]. (2) In Theorems 4 and 5, it is sufficient to assume that S runs over all invertible *hermitean* operators. (3) Theorems 4 and 5 are still valid for any B^* -algebra.

4. In this section we shall give the following theorem:

Theorem 6. *If T is an invertible operator which has the polar decomposition $T = UR$ with the cramped unitary operator U , then*

$$(9) \quad 0 \notin \text{co} \left[\frac{\overline{W(R)}}{\overline{W(U^*)}} \right]$$

and

$$(10) \quad \text{co } \sigma(T) \subset \text{co} \left[\frac{\overline{W(R)}}{\overline{W(U^*)}} \right].$$

To prove the theorem, we need the following well-known theorem:

Theorem 7 (Williams [12]). *If $0 \notin \overline{W(A)}$, then*

$$(11) \quad \sigma(A^{-1}B) \subset \frac{\overline{W(B)}}{\overline{W(A)}}$$

for any $B \in \mathcal{B}(H)$.

Proof of Theorem 6. Since U is cramped by the hypothesis, $\overline{W(U^*)}$ is contained in an open half plane which excludes the origin. Since R is strictly positive, $\overline{W(R)}$ is included in $(0, \infty)$. Hence the right hand side of (9) is included in the open half plane which contains $\overline{W(U^*)}^*$, so that the origin is excluded. This proves the first half of the theorem.

The proof of the second half is essentially identical with that of [5; Theorem 3]. However, for the sake of completeness, we shall describe briefly.

Putting $A=U^*$ and $B=R$ in (11) if $T=UR$ is the polar decomposition of T , we have

$$(12) \quad \sigma(T) \subset \frac{\overline{W(R)}}{\overline{W(U^*)}}$$

Making the convex hulls of the both sides of (12), we have (10) as desired.

Theorem 6 implies the following theorem of Berberian quoted in [8] since $\text{co } \sigma(T)$ excludes the origin by (9) and (10):

Theorem 8 (Berberian). *Let T be an invertible operator such that $U=T(T^*T)^{-1/2}$ is cramped, then $0 \notin \text{co } \sigma(T)$.*

In addition, if T is convexoid, then

$$\text{co } \sigma(T) = \overline{W(T)}$$

by the definition, so that Theorems 6 and 8 imply our previous theorem:

Theorem 9 ([5]). *If T is an invertible convexoid operator such that $U=T(T^*T)^{-1/2}$ is cramped, then $0 \notin \overline{W(T)}$.*

In [5] there are given another geometric properties of the spectra of invertible convexoid operators which have the polar decomposition $T=UR$ with the cramped unitary operator U .

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