64. On the α-Deficiency of Meromorphic Functions under Change of Origin

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1. Introduction. Let $f(z)$ be a transcendental meromorphic function in $|z|<\infty$ of order $\rho, 0\leqslant\rho\leqslant\infty$ and of lower order μ. A real number α is said to be admissible to $f(z)$ if $\alpha=0$ when $\rho=0, 0\leqslant\alpha<\rho$ when $0<\rho<\infty$ and $0<\alpha<\infty$ when $\rho=\infty$. We will use the usual symbols of the Nevanlinna theory of meromorphic functions: $T(r,f)$, $N(r,a,f), \delta(a,f)$ etc. (see [2]).

Now, we have introduced in [3] the following symbols in order to avoid the exceptional set in the second fundamental theorem of Nevanlinna for any admissible α to $f(z)$ and $r_0>0$:

$$T_\alpha(r,r_0,f)=\int_{r_0}^{r} T(t,f)/t^{1+\alpha}dt, \qquad N_\alpha(r,r_0,a,f)=\int_{r_0}^{r} N(t,a,f)/t^{1+\alpha}dt$$

and

$$\delta_\alpha(a,f)=1-\limsup_{r\to\infty}\frac{N_\alpha(r,r_0,a,f)}{T_\alpha(r,r_0,f)},$$

where a is any point on the Riemann sphere, and proved:

1) $T_\alpha(r,r_0,f)$ tends to the infinity monotonously as $r\to\infty$ and

$$\limsup_{r\to\infty}\frac{\log T_\alpha(r,r_0,f)}{\log r}=\begin{cases}\rho-\alpha & \text{for } \rho<\infty\\ \infty & \text{for } \rho=\infty,\end{cases}$$

$$\liminf_{r\to\infty}\frac{\log T_\alpha(r,r_0,f)}{\log r}\begin{cases}\geqslant\max(\mu-\alpha,0) & \text{for } \mu<\infty\\ =\infty & \text{for } \mu=\infty,\end{cases}$$

2) $\delta_\alpha(a,f)$ is independent of the choice of r_0 and for admissible $\beta(>\alpha)$ to $f(z)$

$$\delta(a,f)\leqslant\delta_\alpha(a,f)\leqslant\delta_\beta(a,f)\leqslant 1,$$

3) $\sum_a \delta_\alpha(a,f)\leqslant 2$.

We call $\delta_\alpha(a,f)$ α-deficiency of $f(z)$ at a. It is natural to consider whether the α-deficiency depends on the choice of origin or not as well as the Nevanlinna deficiency. In this note, we will show first that $\delta_\alpha(a,f)$ depends on the choice of origin by using Dugué's example ([1]) used for the case of $\delta(a,f)$, and next give some sufficient conditions under which $\delta_\alpha(a,f)$ is invariant under a change of origin by Valiron's method ([4]).

2. Example. Let

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$$f(z)=(\exp(2\pi i e^{z}-1))/(\exp(2\pi i e^{-z}-1))$$
$$f_h(z)=f(z-h),\ h\neq 0,\ \text{real and } \phi(z)=\exp(2\pi i e^{z}-1).$$

We will prove that $\delta_\alpha(0,f)\neq\delta_\alpha(0,f_h)$ or $\delta_\alpha(\infty,f)\neq\delta_\alpha(\infty,f_h)$ occurs for any $0<\alpha<\infty$. Dugué proved in [1]

4) $N(r,0,f)=N(r,f)$, 5) $n(r,0,f)=n(r,0,\phi)-(1+2[r])$,

6) $T(r,f)\leqslant 2T(r,\phi)+O(1)$ and 7) $\lim_{r\to\infty} N(r,0,f_h)/N(r,f_h)=e^{2h}$.

Further, we note

8) $f(z), f_h(z)$ and $\phi(z)$ are of infinite order and of regular growth,

9) $N(r,0,f), N(r,f), N(r,0,f_h)N(r,f_h)$ and $N(r,0,\phi)$ are of infinite order and of regular growth.

Let α be any positive number. Then it is admissible to $f(z), f_h(z)$ and $\phi(z)$ from 8). By 4), we obtain

$$N_\alpha(r,r_0,0,f)=N_\alpha(r,r_0,f)$$

and so

$$\delta_\alpha(0,f)=\delta_\alpha(\infty,f). \tag{1}$$

From 5) Dugué proved

$$N(r,0,f)\geqslant N(r,0,\phi)-3r,$$

so that we have

$$N_\alpha(r,r_0,0,f)\geqslant N_\alpha(r,r_0,0,\phi)-3r^{1-\alpha}. \tag{2}$$

From 6), we have

$$T_\alpha(r,r_0,f)\leqslant 2T_\alpha(r,r_0,\phi)+O(1). \tag{3}$$

Since $r^{1-\alpha}/N_\alpha(r,r_0,0,\phi)$ tends to zero as $r\to\infty$ by using 9), the following inequality obtained from (2) and (3)

$$\frac{N_\alpha(r,r_0,0,f)}{T_\alpha(r,r_0,f)}\geqslant\frac{N_\alpha(r,r_0,0,\phi)-3r^{1-\alpha}}{2T_\alpha(r,r_0,\phi)+O(1)}$$

reduces to

$$1-\delta_\alpha(0,f)\geqslant\frac{1}{2}\limsup_{r\to\infty}\frac{N_\alpha(r,r_0,0,\phi)}{T_\alpha(r,r_0,\phi)}. \tag{4}$$

Now, $\phi(z)$ has two Nevanlinna deficient values of deficiency 1, that is, $\delta(-1,\phi)=\delta(\infty,\phi)=1$, so that there is no other deficient value. This implies from 2) and 3)

$$\limsup_{r\to\infty}\frac{N_\alpha(r,r_0,0,\phi)}{T_\alpha(r,r_0,\phi)}=1.$$

Using this, from (1) and (4) we obtain

$$\delta_\alpha(0,f)=\delta_\alpha(\infty,f)\leqslant\frac{1}{2}. \tag{5}$$

On the other hand, from 7) and 9), we have

$$\lim_{r\to\infty} N_\alpha(r,r_0,0,f_h)/N_\alpha(r,r_0,f_h)=e^{2h},$$

and so

$$1-\delta_\alpha(0,f_h)=e^{2h}(1-\delta_\alpha(\infty,f_h)). \tag{6}$$

By virtue of (5) and (6), we obtain $\delta_\alpha(0,f)\neq\delta_\alpha(0,f_h)$ or $\delta_\alpha(\infty,f)\neq\delta_\alpha(\infty,f_h)$.

This completes the proof.

3. Sufficient conditions. In this paragraph, we will give some sufficient conditions under which $\delta_\alpha(a, f)$ is invariant under a change of origin.

Theorem. *Let $f(z)$ be a transcendental meromorphic function in $|z|<\infty$. If there is an admissible number α to $f(z)$ such that*

$$\lim_{r\to\infty} \frac{T_\alpha(r+1, r_0, f)}{T_\alpha(r, r_0, f)} = 1,$$

then $\delta_\alpha(a, f)$ is invariant under a change of origin for any a.

Proof. It is proved by Valiron ([4]) that if we put $f(z-c)=f_c(z)$ for any finite complex number c, then

$$n(r-|c|, a, f) \leqslant n(r, a, f_c) \leqslant n(r+|c|, a, f),$$

(7) $$(1-\varepsilon_1)N(r-|c|, a, f) \leqslant N(r, a, f_c) \leqslant (1+\varepsilon_1)N(r+|c|, a, f)$$

and

(8) $$(1-\varepsilon_2)T(r-|c|, f) \leqslant T(r, f_c) \leqslant (1+\varepsilon_2)T(r+|c|, f),$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $r\to\infty$. From (7) and (8), we can verify that both $N_\alpha(r, r_0, a, f)$ and $N_\alpha(r, r_0, a, f_c)$ are bounded or

$$(1-\varepsilon_1')N_\alpha(r-|c|, r_0, a, f) \leqslant N_\alpha(r, r_0, a, f_c) \leqslant (1+\varepsilon_1')N_\alpha(r+|c|, r_0, a, f)$$

and

$$(1-\varepsilon_2')T_\alpha(r-|c|, r_0, f) \leqslant T_\alpha(r, r_0, f_c) \leqslant (1+\varepsilon_2')T_\alpha(r+|c|, r_0, f),$$

where $\varepsilon_1', \varepsilon_2' \to 0$ as $r\to\infty$. From these relations, we obtain by using the hypothesis

$$\limsup_{r\to\infty} \frac{N_\alpha(r, r_0, a, f)}{T_\alpha(r, r_0, f)} = \limsup_{r\to\infty} \frac{N_\alpha(r, r_0, a, f_c)}{T_\alpha(r, r_0, f_c)},$$

which shows the validity of our theorem.

Corollary 1. *If the order ρ of $f(z)$ is finite and*

(9) $$\rho-\mu<1, \qquad \mu: \text{lower order of } f(z),$$

then $\delta_\alpha(a, f)$ is invariant under a change of origin for any admissible α to $f(z)$ and any a.

Proof. Under the condition (9), Valiron ([4]) proved

(10) $$\lim_{r\to\infty} T(r+1, f)/T(r, f) = 1.$$

We can prove easily that for any admissible α to $f(z)$ the relation (10) implies

$$\lim_{r\to\infty} T_\alpha(r+1, r_0, f)/T_\alpha(r, r_0, f) = 1,$$

so that we obtain this corollary from Theorem.

Corollary 2. *If for some $\alpha(0<\alpha<1/2)$ admissible to $f(z)$, which is of finite order ρ and of lower order μ,*

$$K_\alpha(f) = \limsup_{r\to\infty} \frac{N_\alpha(r, r_0, 0, f) + N_\alpha(r, r_0, f)}{T_\alpha(r, r_0, f)} = 0,$$

then $\delta_\alpha(a, f)$ is invariant under a change of origin for any a.

Proof. We have proved in [3] that under the condition of this corollary

$$\rho-\mu\leqslant\alpha<\frac{1}{2},$$

so that from Corollary 1, we obtain the result.

References

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