58. Prime Ideals in the Dual Objects of Locally Compact Groups

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1. Let G be a locally compact group, and Ω be the set of equivalence classes of unitary representations of G, dimensions of which are lower than a sufficiently large fixed cardinal number (for instance the large one of countable infinite or $\dim L^2(G)$). Then we can introduce a product operation $\otimes$ in Ω by the Kronecker product of representations, and the addition operation $\oplus$ in Ω by the direct sum of representations (We allow infinite discrete direct sum). So that, a ring-like structure is given in Ω.

Now we shall call a subset $\mathfrak{J}$ an *ideal* in Ω when

i) *$\mathfrak{J}$ is closed with respect to the operation $\oplus$.*

ii) *If ω is in $\mathfrak{J}$ then any subrepresentation* of ω is in $\mathfrak{J}$.

iii) *For any ω in $\mathfrak{J}$ and any ω_0 in Ω, $\omega_0 \otimes \omega$ is in $\mathfrak{J}$.*

Moreover, we shall call an ideal $\mathfrak{J}$ in Ω is *prime* when

iv) *If ω_1, ω_2 are both disjoint to any representations in $\mathfrak{J}$, in the sense of G. W. Mackey* [1], *then $\omega_1 \otimes \omega_2$ too.*

As is well-known, Kronecker product of any ω in Ω and the regular representation $\mathfrak{R}$ is unitary equivalent to a multiple of $\mathfrak{R}$. So that, the set $\mathfrak{J}_{\mathfrak{R}}$ of classes of all subrepresentations of multiples of $\mathfrak{R}$ gives the smallest non-empty (but in general not prime) ideal in Ω.

On the other hand, in the previous paper [2], we gave examples of non-trivial operator fields $\{T(\omega)\}$ over Ω which commute with the both of operations $\otimes$ and $\oplus$, and $T(\mathfrak{R})=0$ (p. 225, Example 3 and p. 226, Example 5). There exists close connection between such an operator field and non-trivial prime ideal.

The purpose of this paper is to show this connection, and to give an example of non-trivial prime ideal in Ω as an extension of the examples in the paper [2]. And this leads to a new proof of that every unitary irreducible representations of compact group are finite dimensional.

2. Now we shall give the correspondence between non-trivial prime ideals in Ω and a family of non-zero operator fields $\{T(\omega)\}$ over Ω which commute with the both of operations $\otimes$ and $\oplus$ and $T(\mathfrak{R})=0$, under the additional condition, that $T(\omega)^{-1}(0)$ is G-invariant for any ω in Ω.

For given such an operator field $\{T(\omega)\}$, it is easy to see that,

$$\mathfrak{J}=\{\omega\in\Omega/T(\omega)=0\} \tag{1}$$

is a non-trivial prime ideal in Ω (cf. the proof of Lemma 4.4. in [2]).

Conversely, if a non-trivial prime ideal $\mathfrak{J}$ in Ω is given, we can construct an operator field which satisfies (1) as follows. At first, we fix an arbitrary element g in G. For any ω in Ω, we can decompose it as $\omega\sim\omega_1\oplus\omega_2$, where ω_1 is disjoint from any representation in $\mathfrak{J}$. And put

$$T(\omega)=U_g(\omega_1)\oplus 0(\omega_2).$$

From above arguments, $\mathfrak{J}$ contains $\mathfrak{R}$, so it is easily shown that the operator field $\{T(\omega)\}$ over Ω is required one.

3. Let $\mathfrak{J}_F$ be the set of classes of unitary representations which don't contain any finite dimensional subrepresentation as a discrete component. Then,

Theorem. *$\mathfrak{J}_F$ is a prime ideal in Ω.*

Before stating the proof of the theorem, we shall show the followings.

Lemma 1. *If $\omega_1\otimes\omega_2$ has a finite dimensional subrepresentation as a discrete component, then the both of ω_1 and ω_2 have the same properties.*

To prove Lemma 1, we use the result of Lemma 2 which is a special case of Lemma 1.

Lemma 2. *If $\omega_1\otimes\omega_2$ contains the trivial representation* **1** *as a discrete component, then the both of ω_1 and ω_2 have finite dimensional subrepresentations as discrete components.*

Proof. 1) Let the spaces of representations ω_1, ω_2 be $\mathcal{H}_1, \mathcal{H}_2$ respectively. Using G. W. Mackey's construction [1], the space of representation $\omega_1\otimes\omega_2$ can be considered as the space of Hilbert-Schmidt operators A from $\overline{\mathcal{H}}_2$ into $\mathcal{H}_1$, and the representation operator of g is given by $A\to U_g^1 A(U_g^2)^*$, corresponding to the operators U_g^1, U_g^2 of ω_1, ω_2 respectively. ($\overline{\omega}, \overline{U}_g$ mean the adjoint representation of ω, U_g respectively. cf. G. W. Mackey [1].)

2) Let A be the Hilbert-Schmidt operator which corresponds to **1**-component in $\omega_1\otimes\omega_2$ by the assumption, then

$$U_g^1 A(U_g^2)^*=A, \qquad \text{for any } g \text{ in } G. \tag{2}$$

That is, the trace class operator A^*A on $\overline{\mathcal{H}}_2$ satisfies,

$$A^*A(U_g^2)^*=(U_g^2)^*A^*A, \qquad \text{for any } g \text{ in } G. \tag{3}$$

A^*A is not zero, and any eigenspace of A^*A, corresponding to non-zero eigenvalue, is finite dimensional. And it is easy to see from (3) that each eigenspace, is invariant with respect to $(U_g^2)^*$. That is, ω_2 contains a finite dimensional subrepresentation as a discrete component.

3) $\omega_1\otimes\omega_2$ is equivalent to $\omega_2\otimes\omega_1$, so we can exchange the roles of

ω_1 and ω_2, then we obtain the result about ω_1.

Proof of Lemma 1. If $\omega_1 \otimes \omega_2$ contains a finite dimensional subrepresentation D as a discrete component, so $\omega_1 \otimes \omega_2 \otimes \bar{\omega}_1 \otimes \bar{\omega}_2$ contains $D \otimes \bar{D}$ as a discrete component. But from the theory of finite dimensional representations, $D \otimes \bar{D}$ contains the trivial representation **1** as a discrete component. That is, $\omega_1 \otimes \omega_2 \otimes \bar{\omega}_1 \otimes \bar{\omega}_2$ contains the trivial representation as a discrete component.

From the associativity of Kronecker products, we can use the result of Lemma 2 to the pair ω_1 and $\omega_2 \otimes \bar{\omega}_1 \otimes \bar{\omega}_2$, so the required result for ω_1 is given. The result for ω_2 is easily deduced by exchanging the role of ω_1 and ω_2 as above.

Proof of Theorem. Evidently $\mathfrak{J}_F$ satisfies i), ii) and iv). And the property iii) is shown by Lemma 1 directly.

4. Corollary. *Every irreducible unitary representations of compact groups are finite dimensional.*

Proof. It is enough to show that $\mathfrak{J}_F$ is empty set for compact group. Indeed, if not, since $\mathfrak{J}_{\mathfrak{R}}$ is the smallest non-empty ideal, $\mathfrak{J}_F$ must contain $\mathfrak{R}$. But for a compact group, $\mathfrak{R}$ contains trivial representation **1** as a discrete component (in fact, the constant function is in $L^2(G)$). So $\mathfrak{J}_F$ has to contain **1**. This contradicts to the definition of $\mathfrak{J}_F$.

References

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[2] N. Tatsuuma: A duality theorem for locally compact groups. J. Math. Kyôto Univ., **6**, 187–293 (1967).