57. A Geometrical Method for Optimal Control Problem for Some Non-linear Systems

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0. Introduction. In this note we study the problem of optimal control for some non-linear systems.

Let us consider the following control system:

$$\frac{dx}{dt}=f(x,u), \tag{1}$$

where u is a control parameter and belongs to some control domain U. As was shown by E. Roxin [1] and J. Warga [3], it is proper to assume the set $F(x)=\{f(x,u)\,;u\in U\}$ is compact and convex. In fact convexity of $F(x)$ implies the closedness of the reachable set of the system (1), therefore it guarantees the existence of optimal control for most control systems, at least for time-optimal problem. Moreover, for the general control system,

$$\frac{dx}{dt}\in G(x), \tag{2}$$

if we take its relaxed system (3) instead of (2),

$$\frac{dx}{dt}\in \text{Convex hull of } G(x) \tag{3}$$

then for any solution $x(t)$ of (3) there exists a solution of (2) which approximates $x(t)$ uniformely under fairly general condition, and consequently it will be proper to consider the system (3) in the place of (2).

For the simplicity we consider the time-optimal problem and assume $F(x)$ is a compact convex set generated by finitely many extremal points (vectors).

In the problem of time-optimal control, the value $f(x,u)$ itself is more important than the one of control parameter u, so we set the system (1) in the following form:

$$\frac{dx}{dt}\in \text{Convex}\,\{X_1(x),\cdots,X_r(x)\}, \tag{4}$$

where x denotes a point of R^n, $X_i(x)$ $(i=1,\cdots,r)$ smooth vector fields on R^n, and Convex $\{X_1(x),\cdots,X_r(x)\}$ the convex set generated by the points (vectors) $X_1(x),\cdots,X_r(x)$.

1. Definitions. $x=x(t)=x(t\,;x_0,t_0)(x(t_0\,;x_0,t_0)=x_0)$ is said to be an admissible trajectory of the control system (4), when it is piece-wise smooth and satisfies the following relation:

$$\frac{dx(t)}{dt} \in \text{Convex}\{X_1(x(t)), \cdots, X_r(x(t))\} \qquad \text{a.e.t.}$$

For any points x_1 and x_2 in the domain D of R^n, the admissible trajectory $x(t)$ is said to be the trajectory joining x_1 to x_2 in D, when there exist times t_1 and t_2 such that $x(t_1)=x_1$, $x(t_2)=x_2$, $t_1 \leq t_2$ and $x(t) \in D$ for $t_1 \leq t \leq t_2$. We denote the set of all trajectories joining x_1 to x_2 in D by $T(x_1 \rightarrow x_2 ; D)$.

We call the trajectory $\tilde{x}(t) \in T(x_1 \rightarrow x_2 ; D)$ the time-optimal trajectory from x_1 to x_2 in D, if $\tau(\tilde{x}(t) ; x_1 \rightarrow x_2) \leq \tau(x(t) ; x_1 \rightarrow x_2)$ for any $x(t) \in T(x_1 \rightarrow x_2 ; D)$, where $\tau(x(t) ; x_1 \rightarrow x_2) = \min\{t_2 - t_1 ; x(t_1)=x_1, x(t_2)=x_2\}$.

Now the trajectory $x(t) \in T(x_1 \rightarrow x_2 ; D)$ is said to be $(X_{i_1} \rightarrow X_{i_2} \rightarrow \cdots \rightarrow X_{i_r})$-bang-bang type between x_1 and x_2 when there exist $t_0 \leq t_1 \leq \cdots \leq t_r$ such that $x(t_0)=x_1$, $x(t_r)=x_1$, and

$$\frac{dx(t)}{dt} = X_{i_k}(x(t)) \text{ for } t_{k-1} \leq t \leq t_k \quad (k=1, \cdots, r).$$

2. Local structure of optimal trajectory. A point $x \in R^n$ is said to be $(X_{i_1} \rightarrow X_{i_2} \rightarrow \cdots \rightarrow X_{i_r})$-type when there exists a neighbourhood $U(x)$ of x such that, for any two points x_1 and x_2 of $U(x)$ if $T(x_1 \rightarrow x_2 ; U(x)) \neq \phi$ then there exists one and only one time-optimal trajectory in R^n and the trajectory is $(X_{i_1} \rightarrow X_{i_2} \rightarrow \cdots \rightarrow X_{i_r})$-bang-bang type between x_1 and x_2.

Our aim is to give a necessary and sufficient condition which guarantees the point is $(X_{i_1} \rightarrow X_{i_2} \rightarrow \cdots \rightarrow X_{i_r})$-bang-bang type except the set of "singular points".

Now let us assume $n=r=2$ for the simplicity. We give some generalizations and remarks for the general cases.

Theorem.[*)] *Let $X_1(x)$ and $X_2(x)$ be twice-continuously differentiable vector fields in R^2. If $\Delta(X_1, X_2) > 0 [<0]$ at $x \in R^n$, then x is the point of $(X_1 \rightarrow X_2) - [(X_2 \rightarrow X_1) -]$-bang-bang type for the system;*

$$\text{(S)} \qquad \frac{dx}{dt} \in \text{Convex}\{X_1(x), X_2(x)\}.$$

Here $\Delta(X_1, X_2) = \det(X_1, X_2) \det([X_2, X_1], X_1 - X_2)$. $[X_2, X_1] = \nabla_{X_2} X_1 - \nabla_{X_1} X_2$ denotes the commutator product of the vector fields X_2 and X_1, and ∇_X the direction derivative: $\nabla_X = X^1(x^1, x^2)\, \partial/\partial x^1 + X^2(x^1, x^2)\, \partial/\partial x^2$, $X = (X^1(x^1, x^2), X^2(x^1, x^2))$.

3. Proof of the theorem. Lemma. *Let $X_1(x)$ and $X_2(x)$ be differentiable vector fields on R^n, then we have $\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} = \nabla_{[X_1, X_2]}$.*

The proof of this lemma is a direct consequence of the easy computation so we omit it.

Now let us return to the verification of the theorem.

Let us assume $\Delta(X_1, X_2) > 0$ at $x_0 \in R^2$, and consider the following equation:

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*) Added in proof: Compare the method in "Hermes and La Salle: Functional Analysis and Time Optimal Control, Academic Press, pp. 120–128 (1969)".

$$\text{(E)} \qquad \frac{dx}{ds}=X(x)=X_1(x)-X_2(x).$$

Let $f(x)$ be a twice continuously differentiable integral of the equation (E), i.e. $\nabla_X f=0$, such that $\nabla_{X_1} f>0$. Existence of such integral is assumed only on some neighbourhood of x_0.

Let us show that $\nabla_X \nabla_{X_1} f>0$ at x_0:

$$\nabla_X\nabla_{X_1}f=\nabla_X\nabla_{X_1}f-\nabla_{X_1}\nabla_X f=\nabla_{[X,X_1]}f=\nabla_{[X_1-X_2,X_1]}f=\nabla_{[X_1,X_2]}f.$$

Put, $[X_1,X_2]=\alpha X_1+\beta X$, then we can see that $\alpha>0$ at x_0. In fact,

$$\begin{aligned}\Delta(X_1,X_2)&=\det(X_1,X_2)\circ\det([X_2,X_1],X_1-X_2)\\&=\det(X_1,X_2)\circ\det(-\alpha X_1-\beta X,X)\\&=\det(X_1,X_2)\det(-\alpha X_1,X_1-X_2)\\&=\det(X_1,X_2)\det(\alpha X_1,X_2)\\&=\alpha(\det(X_1,X_2))^2>0\end{aligned}$$

$\therefore\ \alpha>0$

Therefore, $\nabla_X\nabla_{X_1}f=\nabla_{\alpha X_1+\beta X}f=(\alpha\nabla_{X_1}+\beta\nabla_X)f=\alpha\nabla_{X_1}f>0.$

Let $U(x_0)$ be a neighbourhood of x_0 such that it is bounded by the four solution-curves of the systems:

$$\text{(E)}_i \qquad \frac{dx}{ds}=X_i(x) \qquad (i=1,2).$$

Such a neighbourhood exists because of the linear independence of $X_1(x)$ and $X_2(x)$. Making $U(x_0)$ small, if necessary, we can assume $\nabla_X\nabla_{X_1}f$ is positive in $U(x_0)$ by the continuity of it.

Let us take points x_1 and x_2 in $U(x_0)$ such that $T(x_1\to x_2;U(x_0))\neq\phi$. Let $\tilde{x}(t)$ be the trajectory joining x_1 to x_2 in $U(x_0)$ of $(X_1\to X_2)$-bang-bang type. Existence and uniqueness of such trajectory is clear for the shape of $U(x_0)$. Let $x(t)$ be any trajectory of $T(x_1\to x_2;U(x_0))$. Because of the linear independence of $X_1(x)$ and $X_2(x)$, the solution curve $x(t)$ of the system (S) transverses to the family of curves defined by $f(x)=A$ ($=$constant) in $U(x_0)$. Therefore, we can define the function $A(t)$ of t depending on $x(t)$ by $A(t)=f(x(t))$.

As $x(t)$ is the solution of (S), so we can write

$$\frac{dx(t)}{dt}=(1-u(t))X_1(x(t))+u(t)X_2(x(t)),$$

where $u(t)$ is a piece-wise continuous function such that $0\leq u(t)\leq 1$.

$$\begin{aligned}\frac{dA}{dt}&=\nabla_{(1-u(t))X_1(x(t))+u(t)X_2(x(t))}f\\&=\nabla_{X_1(x(t))}f-u(t)\nabla_{X_1(x(t))-X_2(x(t))}f=\nabla_{X_1(x(t))}f.\end{aligned}$$

Now

$$\tau(x(t);x_1\to x_2)=\int dt=\int_{A_1=f(x_1)}^{A_2=f(x_2)}\frac{dA}{\nabla_{X_1(x(t))}f},$$

$$\tau(\tilde{x}(t);x_1\to x_2)=\int_{A_1}^{A_2}\frac{dA}{\nabla_{X_1(\tilde{x}(t))}f}.$$

Clearly $\nabla_{X_1(x(t))}f \leq \nabla_{X_1(\tilde{x}(\tilde{t}))}f$ on the curve $f(x)=A$ because $\nabla_X \nabla_{X_1} f>0$ (here, $f(x(t))=f(\tilde{x}(\tilde{t}))=A$). So,

$$\tau(\tilde{x}(t)\,;\,x_1\rightarrow x_2)\leq \tau(x(t)\,;\,x_1\rightarrow x_2),$$

where equality holds if and only if $\tilde{x}(t)=x(t)$ up to time translation. This proves the time-optimality of the trajectory $\tilde{x}(t)$ in $U(x_0)$, and consequently in R^2. q.e.d.

4. Generalization and remarks. It is interesting and important problem to extend these arguments to the more general cases. But it is not possible to extend our results by the same arguments without some additional conditions.

Let us consider the case $n>2, n\geq r$.

Put $Y_i(x)=X_i(x)-X_r(x)$ $i=1,\cdots,r-1$. Assume that for any $x\in R^n$ there exists a $(r-1)$-dimensional submanifold $M^{r-1}(x)$ such that $x\in M^{r-1}(x)$ and the tangent space $T_yM^{r-1}(x)$ to $M^{r-1}(x)$ at any point $y\in M^{r-1}(x)$ is spanned by $\{Y_i(y): i=1,\cdots,r-1\}$. Namely, if $Y_i(x), i=1,\cdots,r-1$ are completely integrable, then the arguments of 3 will be available with slight modification even for this case. In the arguments of section 3, $M^{r-1}(x)$ are the submanifolds $\{f(x)=A\}$.

In the case $r>n$, another serious difficulties arise. They are the very proper difficulties of the control problem. However, by considering the systems

$$\frac{dx}{dt}\in \text{Convex}\,\{X_{i_1},\cdots,X_{i_n}\},\qquad 1\leq i_1\leq i_2\leq\cdots\leq i_n\leq r.$$

all together, we can deduce some information from them for the system considered. For instance, let us consider the system:

$$\frac{dx}{dt}\in \text{Convex}\,\{X_1(x),\cdots,X_3(x)\},\qquad x\in R^2.$$

If we have $\Delta(X_1,X_2)>0, \Delta(X_2,X_3)>0$, and $\Delta(X_1,X_3)>0$ near x_0, then the optimal trajectory is of (X_1,X_2,X_3)-bang-bang type near x_0.

5. Example. For the example, consider Mathien equation which appears in the study of parametric-resonance:

$$\frac{d^2x}{dt^2}=-(1+\varepsilon u)x, \tag{5}$$

where u is a control parameter such that $|u|\leq 1$.

Put $x=x^1, dx/dt=x^2$, then the system (5) becoms

$$\frac{dx^1}{dt}=x^2\qquad \frac{dx^2}{dt}=-(1+\varepsilon u)x^1. \tag{6}$$

If we put $X_1=(x^2, -(1+\varepsilon)x^1)$ and $X_2=(x^2, -(1-\varepsilon)x^1)$, then the system will be the form:

$$\frac{dx}{dt}\in \text{Convex}\,\{X_1,X_2\}\qquad x=(x^1,x^2)\in R^2.$$

In this case, we obtain by easy computation

$$\Delta(X_1, X_2) = 8\,\varepsilon^3 (x^1)^3 x^2.$$

$x^1 = 0$ and $x^2 = 0$ are the sets of singular points.

References

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[5] Pontryagin, *et al.*: The Mathematical Theory of Optimal Processes. Wiley (Interscience) (1962).