56. *Remarks on the Eichler Cohomology of Kleinian Groups*

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1. Let Γ be a finitely generated kleinian group, Ω its region of discontinuity, Λ its limit set and $\lambda(z)|dz|$ the Poincaré metric on Ω. We denote by Δ an arbitrary Γ-invariant union of components of Ω. In this note we assume that Δ/Γ is a finite union of compact Riemann surfaces, and consider relations between the Kra and the Ahlfors decompositions for $H^1(\Gamma, \Pi_{2q-2})$.

2. We fix an integer $q\geqq 2$. Let Ξ be an Γ-module. A mapping $p: \Gamma\to\Xi$ is called Ξ-cocycle if $p_{AB}=p_A\cdot B+p_B$, $A, B\in\Gamma$. If $f\in\Xi$, its coboundary δf is the cocycle $A\to f\cdot A-f$, $A\in\Gamma$. The first cohomology space $H^1(\Gamma, \Xi)$ is the space of cocycles factored by the space of coboundaries. The Γ-modules used in this note are (1) Π_{2q-2}, the vector space of complex polynomials in one variable of degree at most $2q-2$, with $v\cdot A(z)=v(Az)A'(z)^{1-q}$, $v\in\Pi_{2q-2}$ and $A\in\Gamma$ and (2) $\boldsymbol{H}_r(\Delta)(\boldsymbol{M}_r(\Delta))$ the vector space of holomorphic (meromorphic) functions on Δ, with $f\cdot A(z)=f(Az)A'(z)^{1-q}$, $f\in\boldsymbol{H}_r(\Delta)(\boldsymbol{M}_r(\Delta))$, $A\in\Gamma$, where r is an integer. We call $\boldsymbol{H}_r(\Delta,\Gamma)$ and $\boldsymbol{M}_r(\Delta,\Gamma)$, the spaces of holomorphic and meromorphic automorphic forms of weight $(-2r)$ on Δ for Γ, respectively. Two meromorphic (holomorphic) Eichler integrals of order $1-q$ are identified if they differ an element of Π_{2q-2}. This identification space is denoted by $\boldsymbol{E}_{1-q}(\Delta,\Gamma)(\boldsymbol{E}^0_{1-q}(\Delta,\Gamma))$. If $a_1, a_2, \cdots, a_{2q-1}$ are distinct points in Λ and $\phi\in\boldsymbol{H}_q(\Delta,\Gamma)$, then

$$F(z)=\frac{(z-a_1)\cdots(z-a_{2q-1})}{2\pi i}\iint_{\Omega}\frac{\lambda^{2-2q}(\zeta)\bar{\phi}(\zeta)d\zeta\wedge d\bar{\zeta}}{(\zeta-z)(\zeta-a_1)\cdots(\zeta-a_{2q-1})}$$

is a potential for ϕ (Bers [2]). We denote by $\mathrm{Pot}(\phi)$ a potential for ϕ. A mapping $\alpha: \boldsymbol{E}^0_{1-q}(\Delta,\Gamma)\to H^1(\Gamma,\Pi_{2q-2})$ is defined as $\alpha_A(f)=f\cdot A-f$ for $f\in\boldsymbol{E}^0_{1-q}(\Delta,\Gamma)$ and $A\in\Gamma$. A mapping $\beta^*: \boldsymbol{H}_q(\Delta,\Gamma)\to H^1(\Gamma,\Pi_{2q-2})$ is defined by setting $\beta^*_A(\phi)=\mathrm{Pot}(\phi)\cdot A-\mathrm{Pot}(\phi)$ for $\phi\in\boldsymbol{H}_q(\Delta,\Gamma)$.

Theorem A (The Kra decomposition). *Every $p\in H^1(\Gamma,\Pi_{2q-2})$ can be written uniquely as $p=\alpha(f)+\beta^*(\phi)$ with $f\in\boldsymbol{E}^0_{1-q}(\Delta,\Gamma)$ and $\phi\in\boldsymbol{H}_q(\Delta,\Gamma)$.*

3. For $f\in\boldsymbol{E}_{1-q}(\Delta,\Gamma)$, the polynomials $f(Az)A'(z)^{1-q}-f(z)$ are the periods of f, and we write $f(Az)A'(z)^{1-q}-f(z)=pd_Af(z)$. The periods determine a canonical isomorphism $pd: \boldsymbol{E}_{1-q}(\Delta,\Gamma)\to H^1(\Gamma,\Pi_{2q-2})$. Thus pdf, $f\in\boldsymbol{E}_{1-q}(\Delta,\Gamma)$, is a cohomology class and $pd\boldsymbol{E}_{1-q}(\Delta,\Gamma)$ is the image

of $\boldsymbol{E}_{1-q}(\Delta, \Gamma)$ under the period mapping.

Theorem B (The Ahlfors theorem).

$$H^1(\Gamma, \Pi_{2q-2}) = pd\boldsymbol{E}_{1-q}(\Delta, \Gamma).$$

Let $\Delta/\Gamma = S_1 \cup \cdots \cup S_n$, S_i being compact Riemann surfaces. Choose a point $\zeta_j \in \Pi^{-1}(S_j)$ which is not an elliptic fixed point nor a q-Weierstrass point, where $\Pi: \Delta \to \Delta/\Gamma$ is the natural projection mapping. Let $d_j = \dim(\boldsymbol{H}_q(\Delta,\Gamma) \mid \Pi^{-1}(S_j))$. Set $g(z,\zeta) = \sum_{A\in\Gamma} (z - A\zeta^{-1}A'(\zeta)^q, z \in \Omega, \zeta \in \Omega$. Here we may assume without any loss of generality that $\infty \notin \Lambda$, and ∞ is not elliptic fixed point. Define $g_\nu(z,\zeta) = \partial^{\nu-1}g/\partial\zeta^{\nu-1}$, and set $G_\nu(z) = g_\nu(z, \zeta_j)$, $\nu = 1, 2, \cdots, d$ (Ahlfors [1] and Kra [7]). We denote by $\tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma)$ the space spaned by $G_\nu(z)$, $\nu = 1, \cdots, d$; $j = 1, 2, \cdots, n$. Then Theorem B implies

Theorem B′ (cf. Kra [7]).

$$H^1(\Gamma, \Pi_{2q-2}) = pd\boldsymbol{E}^0_{1-q}(\Delta, \Gamma) + pd\tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma).$$

Let $g \in \tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma)$. We set $\tilde{\beta}^*_A(g) = g(Az)A'(z)^{1-q} - g(z)$ for all $A \in \Gamma$.

4. If Γ is a fuchsian group on the upper half plane U of the first kind without parabolic elements, a mapping $\delta^*: \boldsymbol{H}_q(U, \Gamma) \to H^1(\Gamma, \Pi_{2q-2})$ is defined as

$$\delta^*_A(\phi) = \frac{1}{(2q-2)!}\int_{A^{-1}z_0}^{z_0} (z-\zeta)^{2q-2}\phi(\zeta)d\zeta, \ \phi \in \boldsymbol{H}_q(U, \Gamma) \quad \text{and} \quad A \in \Gamma,$$

where $z_0 \in U$. Then by a similar method as Gunning [5] we have

Theorem 1. (1) *Let Γ be a finitely generated kleinian group with Δ/Γ is a finite union of compact Riemann surfaces. We take $p \in H^1(\Gamma, \Pi_{2q-2})$. Let $p = \alpha(f_1) + \beta^*(\phi)$, $f \in \boldsymbol{E}^0_{1-q}(\Delta, \Gamma)$ and $\phi \in \boldsymbol{H}_q(\Delta, \Gamma)$, and $p = \alpha(f_2) + \tilde{\beta}^*(g)$, $f \in \boldsymbol{E}^0_{1-q}(\Delta, \Gamma)$ and $g \in \tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma)$ be the Kra and the Ahlfors decompositions, respectively, then*

$$(\psi, \phi) = 2\pi i \operatorname{Res}(g\psi) \qquad \textit{for any } \psi \in \boldsymbol{H}_q(\Delta, \Gamma),$$

where we define (ψ, ϕ) as $(\psi, \phi) = 1/2\pi i \iint_{\Delta/\Gamma} \lambda(\zeta)^{2-2q}\psi(\zeta)\overline{\phi(\zeta)}d\zeta \wedge d\bar{\zeta}$.

(2) *In particular when Γ is a fuchsian group of the first kind without parabolic elements, then*

$$\alpha_A(f) = (2q-2)!\, \delta^*_A(D^{2q-1}f) \qquad \textit{for } f \in \boldsymbol{E}^0_{1-q}(\Delta, \Gamma) \textit{ and } A \in \Gamma.$$

Proof. (1) $(f_1 - f_2)(Az)A'(z)^{1-q} - (f_1 - f_2)(z) + \operatorname{Pot}(\phi)(Az)A'(z)^{1-q} - \operatorname{Pot}(\phi)(z) = g(Az)A'(z)^{1-q} - g(z)$ for all $A \in \Gamma$, so that $h(Az)A'(z)^{1-q} = h(z)$ for all $A \in \Gamma$, where we set $f_1 - f_2 = f \in \boldsymbol{E}^0_{1-q}(\Delta, \Gamma)$ and $h = g - f - \operatorname{Pot}(\phi)$. By definition, for any $\psi \in \boldsymbol{H}_q(\Delta, \Gamma)$,

$$(\psi, \phi) = \iint_\omega \lambda^{2-2q}(z)\psi(z)\overline{\phi(z)}dz \wedge d\bar{z}$$
$$= \iint_\omega \psi(z)dz \wedge \bar{\partial}(\operatorname{Pot}(\phi)(z)),$$

which does not depend on the choice of ω, where ω is a fundamental region in Δ for Γ, that is, $\omega = \bigcup_{i=1}^n (\omega_i \cap \Delta)$, ω_i is a fundamental region in Δ_i for Γ_i, since $\Delta/\Gamma = \bigcup_{i=1}^n \Delta_i/\Gamma_i$ (Bers [2]). Applying Stokes' theorem we have

$$(\psi, \phi)=\int_{\partial\omega} \operatorname{Pot}(\phi)(z)\psi(z)dz$$
$$=\int_{\partial\omega} g(z)\psi(z)dz-\int_{\partial\omega} h(z)\psi(z)dz-\int_{\partial\omega} f(z)\psi(z)dz$$
$$=2\pi i \operatorname{Res}(g\psi),$$

because arcs of $\partial\omega$ are identified in pairs by elements of Γ and in view of $h\psi dz$ is Γ-invariant, so that $\int_{\partial\omega} h\psi dz=0$, and $f\psi$ is holomorphic function in ω, so that $\operatorname{Res}(f\psi)=0$.

(2) $$\delta^*(D^{2q-1}f)=1/(2q-2)!\int_{A^{-1}z_0}^{z_0}(z-\zeta)^{2q-2}D^{2q-1}f(\zeta)d\zeta$$
$$=1/(2q-2)!\int_{z_0}^{Az}(Az-\zeta)^{2q-2}D^{2q-1}f(\zeta)d\zeta\times A'(z)^{1-q}$$
$$-1/(2q-2)!\int_{z_0}^{z}(z-\zeta)^{2q-2}D^{2q-1}f(\zeta)d\zeta.$$

Set $h(z)=1/(2q-2)!\int_{z_0}^{z}(z-\zeta)^{2q-2}D^{2q-1}f(\zeta)d\zeta$. Then $h\in \boldsymbol{E}_{1-q}^0(U,\Gamma)$. In fact, first, $h(Az)A'(z)^{1-q}-h(z)=1/(2q-2)!\int_{A^{-1}z_0}^{z_0}(z-\zeta)^{2q-2}D^{2q-1}f(\zeta)d\zeta$ $=v(z)$, $v\in \Pi_{2q-2}$. Secondly, setting $D^{2q-1}h=\phi$, we have by the Cauchy formula

$$\phi(Az)A'(z)^q=(2q-1)!/2\pi i\int_{\zeta\in C_{Az}} h(\zeta)/(\zeta-Az)^{2q}d\zeta\times A'(z)^q$$
$$=\frac{(2q-1)!}{2\pi i}\int_{w\in C_z}\frac{h(Az)}{(Aw-Az)^{2q}}dAw\times A'(z)^q$$
$$=\frac{(2q-1)!}{2\pi i}\left(\int_{w\in C_z}\frac{h(w)dw}{(w-z)^{2q}}+\int_{w\in C_z}\frac{v(w)}{(w-z)^{2q}}dw\right),$$

where C_z and C_{Az} are small circles about the points z and Az, respectively. Since $v\in \Pi_{2q-2}$, $\int_{w\in C_z} v(w)/(w-z)^{2q}dw=0$ and hence

$$\phi(Az)A'(z)^q=\frac{(2q-1)!}{2\pi i}\int_{w\in C_z}\frac{h(w)}{(w-z)^{2q}}dw=\phi(z).$$

On the other hand we see easily $D^{2q-1}h(z)=D^{2q-1}f(z)$, and hence $h=f+v_1$, $v_1\in \Pi_{2q-2}$. Thus we have

$$(2q-2)!\,\delta_A^*(D^{2q-1}f)(z)=h(Az)A'(z)^{1-q}-h(z)$$
$$=f(Az)A'(z)^{1-q}-f(z)+v_1(Az)A'(z)^{1-q}-v_1(z)=\alpha_A(f)(z)$$

for all $A\in\Gamma$. Our proof is now complete.

Remark. It is easy to see by modifying the above proof that Theorem 1 (2) is satisfied in the case of which Γ is a finitely generated kleinian group without parabolic elements with an simply connected invariant component Δ.

5. We denote by $H_0^1(\Gamma, \Pi_{2q-2})$ the subspace of $H^1(\Gamma, \Pi_{2q-2})$ each element p of which has decompositions $p=\alpha(f)+\beta^*(\phi)=\alpha(f)+\tilde{\beta}^*(g)$, $f\in \boldsymbol{E}_{1-q}^0(\Delta,\Gamma)$, $\phi\in \boldsymbol{H}_q(\Delta,\Gamma)$ and $g\in \tilde{\boldsymbol{E}}_{1-q}(\Delta,\Gamma)$.

Theorem 2. *If Γ is a kleinian group with an invariant region Δ and Δ/Γ is compact, then*

$$\dim \boldsymbol{H}_q(\Delta, \Gamma) \leqq \dim H_0^1(\Gamma, \Pi_{2q-2}) \leqq 2 \dim \boldsymbol{H}_q(\Delta, \Gamma).$$

The first inequality becomes equality when Γ is a Schottky group, and the second one is attained in the case that Γ is a fuchsian group of the first kind without parabolic elements.

Lemma 1 (Bers [2]). *Let Γ be a fuchsian group of the first kind without parabolic elements. Then for any $p \in \alpha(\boldsymbol{E}_{1-q}^0(U, \Gamma))$, there exists a $\phi \in \boldsymbol{H}_q(L, \Gamma)$ such that $\beta^*(\phi)=p$, and for any $p \in \beta^*(\boldsymbol{H}_q(U, \Gamma))$, there exists an $f \in \boldsymbol{E}_{1-q}^0(L, \Gamma)$ such that $\alpha(f)=p$, where L is the lower half plane.*

The proof of Theorem 2. The second inequality is obvious from $\dim H(\Gamma, \Pi_{2q-2}) \leqq 2 \dim \boldsymbol{H}_q(\Delta, \Gamma)$ (see Kra [6]).

Let $d=\dim \boldsymbol{H}_q(\Delta, \Gamma)$ and $e=\dim \boldsymbol{E}_{1-q}^0(\Delta, \Gamma)$. First we remark that $d \geqq e$ (Kra [6]). Let $\phi_1, \phi_2, \cdots, \phi_d$ be a basis of $\boldsymbol{H}_q(\Delta, \Gamma)$ and $f_1, f_2, \cdots, f_e$ be that of $\boldsymbol{E}_{1-q}^0(\Delta, \Gamma)$. Then $\beta^*(\phi_1), \beta^*(\phi_2), \cdots, \beta^*(\phi_d)$ is a basis of $\beta^*(\boldsymbol{H}_q(\Delta, \Gamma))$ and $\alpha(f_1), \alpha(f_2), \cdots, \alpha(f_e)$, is that of $\alpha(\boldsymbol{E}_{1-q}^0(\Delta, \Gamma))$. Hence we have

$$\begin{cases} \beta^*(\phi_1)=a_{11}\alpha(f_1)+\cdots+a_{1e}\alpha(f_e)+\tilde{\beta}^*(g_1) \\ \beta^*(\phi_2)=a_{21}\alpha(f_1)+\cdots+a_{2e}\alpha(f_e)+\tilde{\beta}^*(g_2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta^*(\phi_d)=a_{d1}\alpha(f_1)+\cdots+a_{de}\alpha(f_e)+\tilde{\beta}^*(g_d) \end{cases}$$

where $a_{ij} \in \boldsymbol{C}$ $(i=1, 2, \cdots, d; j=1, 2, \cdots, e)$.

$$\operatorname{Rank}\begin{pmatrix} a_{11} & a_{12}\cdots a_{1e} \\ a_{21} & a_{22}\cdots a_{2e} \\ \cdots & \cdots \\ a_{d1} & a_{d2}\cdots a_{de} \end{pmatrix} \leqq e$$

Therefore there are linear independent $\phi'_1, \phi'_2, \cdots, \phi'_{d-e}$ and $g'_1, g'_2, \cdots, g'_{d-e}$ with $\phi'_i \in \boldsymbol{H}_q(\Delta, \Gamma)$ and $g'_i \in \tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma)$ $(i=1, 2, \cdots, d-e)$ such that $\beta^*(\phi'_i) = \tilde{\beta}^*(g'_i)$. Since $\alpha(\boldsymbol{E}_{1-q}^0(\Delta, \Gamma)) \subset H_0^1(\Gamma, \Pi_{2q-2})$, we have

$$\begin{aligned} \dim H_0^1(\Gamma, \Pi_{2q-2}) &\geqq \dim \alpha(\boldsymbol{E}_{1-q}^0(\Delta, \Gamma))+d-e \\ &=e+d-e=d. \end{aligned}$$

If Γ is a Schottky group, then $\alpha(\boldsymbol{E}_{1-q}^0(\Delta, \Gamma))=0$ and $\tilde{\beta}^*(\tilde{\boldsymbol{E}}_{1-q}(\Delta, \Gamma)) = H^1(\Gamma, \Pi_{2q-2}) = \beta^*(\boldsymbol{H}_q(\Delta, \Gamma))$, so that $\dim H_0^1(\Gamma, \Pi_{2q-2}) = \dim H^1(\Gamma, \Pi_{2q-2}) = \dim \boldsymbol{H}_q(\Delta, \Gamma)$. Let Γ be a fuchsian group of the first kind without parabolic elements. First we show that for any $p \in \tilde{\beta}^*(\boldsymbol{E}_{1-q}(U, \Gamma))$ there exists a function $f \in \boldsymbol{E}_{1-q}^0(U, \Gamma)$ such that $\alpha(f)=p$. Set $p=\tilde{\beta}^*(g)$. From the construction of the functions g_i, $i=1, 2, \cdots, d$, we see that the function g is holomorphic on L and satisfies $g(Az)A'(z)^{1-q}-g(z)=p_A(z)$, $z \in L$ for all $A \in \Gamma$. Thus $g \in \boldsymbol{E}_{1-q}^0(L, \Gamma)$. Hence $p=\alpha_L(g)$, where α_L denote an operator of α in L. From Lemma 1 there exists $\phi \in \boldsymbol{H}_q(U, \Gamma)$ such that $p=\alpha_L(g)=\beta_U^*(\phi)$, which shows $\tilde{\beta}^*(\tilde{\boldsymbol{E}}_{1-q}(U, \Gamma)) \subset \beta^*(H_q(U, \Gamma))$ where

β_U^* means an operator of β^* in U. As we saw in § 4, $\dim \alpha(\boldsymbol{E}_{1-q}^0(U, \Gamma)) = \dim \boldsymbol{H}_q(U, \Gamma)$, but the latter is equal to $\dim \beta^*(\boldsymbol{H}_q(U, \Gamma))$, so that $\tilde{\beta}^*(\tilde{\boldsymbol{E}}_{1-q}(U, \Gamma)) = \beta^*(\boldsymbol{H}_q(U, \Gamma))$. If $p = \alpha(f_1) + \beta^*(\phi) = \alpha(f_2) + \tilde{\beta}^*(g)$ with $f_1, f_2 \in \boldsymbol{E}_{1-q}^0(U, \Gamma)$, $\phi \in \boldsymbol{H}_q(U, \Gamma)$ and $g \in \tilde{\boldsymbol{E}}_{1-q}(U, \Gamma)$, then $\alpha(f_1) = \alpha(f_2)$, that is, $f_1 = f_2$ and $\beta^*(\phi) = \tilde{\beta}^*(g)$. Thus $H_0^1(\Gamma, \Pi_{2q-2}) = H^1(\Gamma, \Pi_{2q-2})$. In conclusion, from $\dim H^1(\Gamma, \Pi_{2q-2}) = 2 \dim \boldsymbol{H}_q(U, \Gamma)$ (see Kra [6]) we have $\dim H_0^1(\Gamma, \Pi_{2q-2}) = 2 \dim \boldsymbol{H}_q(U, \Gamma)$. Our proof is now complete.

From Theorem 1 (2) and Theorem 2 we have the following

Corollary 1. *Let Γ be a fuchsian group of the first kind without parabolic elements. If $p \in H^1(\Gamma, \Pi_{2q-2})$ is represented as $p = \alpha(f_1) + \beta^*(\phi)$ with $f_1 \in \boldsymbol{E}_{1-q}^0(U, \Gamma)$ and $\phi \in \boldsymbol{E}_q(U, \Gamma)$, and $p = (2q-2)!\, \delta^*(D^{2q-1} f_2) + \tilde{\beta}^*(g)$ with $f_2 \in \boldsymbol{E}_{1-q}^0(U, \Gamma)$ and $g \in \tilde{\boldsymbol{E}}_{1-q}(U, \Gamma)$, then $f_1 = f_2$ and $\beta^*(\boldsymbol{H}_q(U, \Gamma)) = \tilde{\beta}^*(\tilde{\boldsymbol{E}}_{1-q}(U, \Gamma))$.*

The following corollary is obtained by the method of Kra [6] and Corollary 1 of Theorem 2.

Corollary 2. *Let Γ be a fuchsian group of the first kind without parabolic elements. If both holomorphic part and meromorphic part of an Eichler integral f have real periods for all elements in Γ then $pdf = 0$, that is, $f_1 = f + v$, $f_1 \in \boldsymbol{M}_q(U, \Gamma)$ and $v \in \Pi_{2q-2}$. Here pdf is real means that for every $A \in \Gamma$, the all coefficients of pdf are real.*

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