85. Remarks on Hypoellipticity of Degenerate Parabolic Differential Operators

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§1. **Introduction.** We have discussed in [2] the hypoellipticity of linear partial differential operators of the form

$$(1)\qquad P=\frac{\partial}{\partial t}+L(t,x;D_x),\qquad x=(x_1,\cdots,x_n)\in R^n,$$

where $D_x=(-i\partial/\partial x_1,\cdots,-i\partial/\partial x_n)$ and $L(t,x;\xi)$ is a polynomial in $\xi\in R^n$ of order 2μ with coefficients in $C^\infty(R_t\times R_x^n)$. In particular we have been interested in operators which are called to be of Fokker-Plank type. These were transformed by a change of independent variable into one having properties (O), (I), (II) and (III) stated in Proposition 1 and Remark of [2] (see also Theorem 3 in §2), and we could show that if an operator possesses these properties, it has a very regular right-parametrix (see Theorem 3 in §2) and hence its transpose is hypoelliptic. Applying this theorem with $I=[-1,1]$ and $\Delta=\{(t,s);\,-1\leqslant s<t\leqslant 1\}$, we can prove, for example, the following

Theorem 1. *Let, for real r, $\langle r\rangle$ be an integer such that $r\le\langle r\rangle<r+1$ and $M_j(t,x;\xi)$ a polynomial in $\xi\in R^n$ of homogeneous order j with coefficients in $C^\infty(R_t\times R_x^n)$. Then both the operator*

$$(2)\qquad P=\frac{\partial}{\partial t}+\sum_{j=0}^{2\mu}t^{\langle jl/2\mu\rangle}M_j(t,x;D_x),\qquad l=0,1,\cdots,$$

and its transpose ${}^tP$ are hypoelliptic in $R^{n+1}=R_t\times R_x^n$, if l is even and if for every compact set K of R^{n+1} there exists a constant $\delta>0$ such that

$$(3)\qquad \operatorname{Re}M_{2\mu}(t,x;\xi)\ge\delta|\xi|^{2\mu},\qquad (t,x)\in K,\ \xi\in R^n.$$

For the proof we use (9) with $t\in[-1,1]$ and (t,s), $-1\le s<t\le 1$, and Lemmas 1 and 2 in §4.

On the other hand Kannai proved recently in [1] that the operator

$$\frac{\partial}{\partial x}-xD_y^2,\qquad D_y=-i\frac{\partial}{\partial y}$$

is hypoelliptic in the plane and moreover its transpose

$$-\frac{\partial}{\partial x}-xD_y^2$$

is not locally solvable there, of course not hypoelliptic. As an extension of this result we can give

Theorem 2. *The transpose of operator* (2), ${}^tP$, *with odd l is* *hypoelliptic in* R^{n+1}, *if condition* (3) *is satisfied for every compact set* K *of* R^{n+1}. *Moreover, in case the coefficients of* P *are independent of* x, P *is not hypoelliptic there.*

This is a corollary of Theorem 4 which is stated in § 2 and whose proof will be completed in § 3 by using Theorem 3 in § 2 and the reasoning adapted in [1]. The proof of Theorem 2 will be briefly given in § 4.

§ 2. Statement of the main theorems. The following theorem is an amelioration of one given in [2].

Theorem 3. *Suppose that the* $L(t,x;D_x)$ *in operator* P *of the form* (1) *possesses real n-square matrices* Γ_t *and* $T_{(t,s)}$ *with entries in* $C^0(I)$ *and* $C^0(\Delta)$, *respectively, which have the following properties* ($I=[0,1]$ *and* $\Delta=\{(t,s); 0\le s<t\le 1\}$):

(O) *There exists a constant* $\nu>0$ *such that* $(t-s)^\nu \|T_{(t,s)}\|$[1] *is bounded in* Δ.

(I) *If* $L_0(t,x;\xi)$ *denotes the leading part of* $L(t,x;\xi)$, *then for every compact domain* Ω *of* R^n *there exists a positive constant* δ *such that, for every* $(t,s)\in\Delta$ *and* $x\in\Omega$,

$$\mathrm{Re}\int_s^t L_0(\tau,x;T_{(t,s)}\xi)d\tau \ge \delta|\xi|^{2\mu}, \qquad \xi\in R^n.$$

(II) *Let* Ω *be an arbitrary compact domain of* R^n. *Then the coefficients of the polynomial in* ξ,

$$\int_s^t L(\tau,x;T_{(t,s)}\xi)d\tau,$$

are all bounded in $\Delta\times\Omega$.

(III) *The* $L(t,x;\xi)$ *is written as a polynomial of* $\Gamma_t\xi$ *with coefficients in* $\mathcal{E}^0(I)(\mathcal{E}(R^n))$[2] *and the inequality*

$$|\Gamma_s\xi|\le \text{const.}\ |\Gamma_{(t,s)}\xi|^{3)}, \qquad \xi\in R^n,$$

is valid for every $(t,s)\in\Delta$, *if we put*

$$\Gamma_{(t,s)}=(t-s)^{-1/2\mu}T^{-1}_{(t,s)}.$$

Then, for each $x_0\in R^n$, *there exist an open neighborhood* V *of* x_0 *and two sequences of distributions on* $W=((-1,1)\times V)\times([0,1)\times V)$,

$$\{E^{(p)}(t,x;s,y)\},\ \{R^{(p)}(t,x;s,y)\} \qquad (p=1,2,\cdots,),$$

such that $E^{(p)}=0$ *and* $R^{(p)}=0$ *for every* p *and for* $t<s$, *satisfying the following, for every* p,

(P. 1) $$P_{(t,x)}E^{(p)}=\delta(t-s)\times\delta(x-y)-R^{(p)},$$

(P. 2) $$E^{(p)}\in C^\infty(W-\{(t,x;s,y);(t,x)=(s,y)\}),$$

(P. 3) *for every* $\varphi(s,y)\in C_0^\infty((0,1)\times V)$

$$\langle E^{(p)},\varphi\rangle_{(s,y)}\in C^\infty((-1,1)\times V),$$

(P. 4) *for every* $\psi(t,x)\in C_0^\infty((-1,1)\times V)$

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1) By $\|T\|$ we denote supremum of the set $\{T\xi; |\xi|=1\}$.

2) $a(t,x)\in\mathcal{E}^0(I)(\mathcal{E}(R^n))$ means that the mapping $t\to a(t,x)\in\mathcal{E}(R^n)$ is continuous in I.

3) We wrote in [2] as $\|\Gamma_s\|\le$ const. $\|\Gamma_{t,s}\|$, but it is not sufficient.

$$\langle E^{(p)}, \psi\rangle_{(t,x)} \in C_0^\infty([0,1)\times V),$$

(P. 5) *for any integer* $N>0$, *there exists an integer* $p_0>0$ *such that*

$$R^{(p)} \in C^N(W) \qquad \textit{for all } p\geq p_0.$$

Their two sequences of distributions on W, $\{E^{(p)}\}$ and $\{R^{(p)}\}$, are called *a very regular right-parametrix* in W of P. The proof of Theorem 3 has been essentially established in [2]. We would make an additional remark that the property (III) can be dropped in case the coefficients of L are independent of x.

Before ending this section we state the main theorem in this note:

Theorem 4. *Suppose that the* $L(t,x;D_x)$ *in operator* P *of the form* (1) *and* $-L(-t,x;D_x)$ *both satisfy the hypothesis of Theorem* 3. *Then* ${}^tP$ *is hypoelliptic in* R^{n+1}.

This will be proved in the next section

§3. Proof of Theorem 4. We give in this section the proof of Theorem 4. Throughout this section we denote by P an operator satisfying the condition mentioned in Theorem 4. It has been established in [2] that ${}^tP$ is hypoelliptic in $(R-\{0\})\times R^n$. Therefore, for the proof of hypoellipticity of ${}^tP$ in R^{n+1}, it suffices to show that ${}^tP$ is hypoelliptic in $(-1,1)\times R^n$.

First, it follows from Theorem 3 that P has a very regular right-parametrix in W satisfying (P. 1)$\sim$(P. 5), since $L(t,x;D_x)$ satisfies the hypothesis in Theorem 3. Let V be an open set stated in Theorem 3, $G=(-1,1)\times V$ and u be a distribution on G satisfying ${}^tPu\in C^\infty(G)$. Taking two domains G_1, G_2 and a function $\beta\in C_0^\infty(G)$ so that $G_1\subset\bar{G}_1\subset G_2\subset\bar{G}_2\subset G$ and $\beta=1$ on G_2, we have

$${}^tP(\beta u)=\beta{}^tPu+X,$$

where $\beta{}^tPu$ is in $C_0^\infty(G)$, and X is a distribution on G with compact support and vanishes on G_2. It then follows from (P. 2) and (P. 4) that

$$(4) \qquad \langle E^{(p)}(t,x;s,y), {}^tP(\beta u)\rangle_{(t,x)} \in C^\infty(G_1\cap([0,1)\times V)).$$

By (P. 1) and (P. 3) we have

$$(5) \qquad (\beta u)(s,y)=\langle E^{(p)}, {}^tP(\beta u)\rangle_{(t,s)}+\langle R^{(p)}, \beta u\rangle_{(t,x)}$$

for all p and $s>0$. On the other hand we can assert by (P. 5) that for any integer $N>0$, there exists an integer $p_1>0$ such that

$$\langle R^{(p)}, \beta u\rangle_{(t,x)} \in C^N([0,1)\times V)$$

for all $p\geq p_1$. Thus we finally obtain by (4) that the right hand side of (5) is in $C^N(G_1\cap([0,1)\times V))$ for all $p\geq p_1$. So that u is infinitely differentiable in $G_1\cap([0,1)\times V)$ and hence u is in $C^\infty([0,1)\times V)$. It follows similarly from the assumption on $-L(-t,x;D_x)$ that u is also in $C^\infty((-1,0]\times V)$.

By the same argument as in [1] we can see that u is in $C^\infty(G)$. In fact, let $\tilde{u}$ be a distribution on G defined by

$$\langle \tilde{u}, \varphi \rangle = \left(\int_0^1 \int + \int_{-1}^0 \int \right) u(t, x) \varphi(t, x) dt dx, \qquad \varphi \in C_0^\infty(G).$$

Set $v = u - \tilde{u}$. Obviously supp $[v]$ is on the hyperplane $t=0$. Therefore, denoting by V_0 a compact subdomain of V, we can find a finite number of distributions on V_0, v_j $(j=1, \cdots, N)$, such that

$$(6) \qquad v = \sum_{j=0}^{N} (Ev_j) \left(\frac{\partial}{\partial t} \right)^j \quad \text{on} \quad (-1, 1) \times V_0,$$

where Ev_j are distributions on $(-1, 1) \times V_0$ defined by

$$\langle Ev_j, \varphi(t, x) \rangle = \langle v_j, \varphi(0, x) \rangle, \qquad \varphi \in C_0^\infty((-1, 1) \times V_0).$$

Calculating we obtain

$$(7) \qquad {}^tPv = (Ev_N) \left(\frac{\partial}{\partial t} \right)^{N+1} + \sum_{j=0}^{N} (Ew_j) \left(\frac{\partial}{\partial t} \right)^j,$$

w_j being some distributions on V_0. On the other hand, we can immediately obtain

$$\langle {}^tP\tilde{u}, \varphi \rangle = \langle {}^tPu - E[u(+0, x) - u(-0, x)], \varphi \rangle$$

for $\varphi \in C_0^\infty((-1, 1) \times V_0)$. Hence

$$(8) \qquad {}^tPv = E[u(+0, x) - u(-0, x)].$$

Thus it follows from (6), (7) and (8) that $v=0$ and hence $\mu = \tilde{u}$. Therefore, by (8) we have $u(+0, x) = u(-0, x)$. Consequently $u \in C^0(G)$. Now, taking account of the fact that ${}^tP(\partial u / \partial t) \in C^0(G)$, we can assert, by the same argument as above, $\partial u / \partial t(+0, x) = \partial u / \partial t(-0, x)$ and so on. This completes the proof of Theorem 4.

§ 4. Proof of Theorem 2. We are going to prove Theorem 2. It is assumed that P is written in the form (2) with odd l and satisfies the hypothesis in Theorem 2. For the proof, we have only to show that

$$L(t, x; D_x) = \sum_{j=0}^{2\mu} t^{\langle jl/2\mu \rangle} M_j(t, x; D_x)$$

and $-L(-t, x; D_x)$ satisfy the hypothesis of Theorem 3. To do so, we have only to choose

$$(9) \qquad \begin{aligned} \Gamma_t &= (t^l)^{1/2\mu} I_n, & t \in [0, 1], \\ T_{(t,s)} &= \left(\frac{l+1}{t^{l+1} - s^{l+1}} \right)^{1/2\mu} I_n, & 0 \leq s < t \leq 1, \end{aligned}$$

where I_n is the identity matrix of order n. In fact these matrices have the properties (O), (I), (II) and (III) in Theorem 3. This can be verified by using the following two lemmas.

Lemma 1. *Let α be real and $\alpha \geq 1$. Then we have*

$$\frac{(x-y)^\alpha}{x^\alpha - y^\alpha} \leq 1 \qquad \text{for } 0 \leq y < x.$$

Lemma 2. *For any integer $l \geq 0$, there exists a constant $C_l > 0$ such that*

$$s^l \leq C_l \frac{t^{l+1} - s^{l+1}}{t - s}$$

for $t > s$ in case l is even and for $t > s \geq 0$ in case l is odd.

Thus it follows from Theorem 4 that ${}^tP$ is hypoelliptic in R^{n+1}.

The latter half of Theorem 2 is showed as follows. Let the coefficients of the P be independent of x. For every $\xi \in R^n$, we introduce, as in [1], functions $u_\xi(t,x)$ defined in $(-1,1)\times R^n$ as

$$u_\xi(t,x)=\exp\left\{ix\xi-\sum_{j=0}^{2\mu}\int_0^t \tau^{\langle jl/2\mu\rangle}M_j(\tau\,;\,\xi)\,d\tau\right\}.$$

Obviously, these are solutions of $Pu=0$. It now follows from (I), (II) and the $T_{(t,s)}$ in (9) that there exist positive constants c and C such that

$$|u_\xi(t,x)|\le C\exp\{-c\,|T^{-1}_{(t,0)}\xi|^{2\mu}\},\qquad \xi\in R^n,$$

for every $(t,x)\in(-1,1)\times R^n$. Thus, if we take a real number s so that $s>2\mu+n$, the function determined by

$$u(t,x)=\int(1+|\xi|)^{-s}u_\xi(t,x)d\xi$$

satisfies the equation $Pu=0$ but is not infinitely differentiable in $(-1,1)\times R^n$. This shows that P is not hypoelliptic in R^{n+1}.

References

[1] Y. Kannai: An unsolvable hypoelliptic differential operator (preprint).
[2] Y. Kato: The hypoellipticity of degenerate parabolic differential operators. J. Funct. Anal., 7, 116–131 (1971).