81. *Complex Powers of a System of Pseudo-differential Operators*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

§0. Introduction. In this paper we shall give complex powers of a system of pseudo-differential operators which is not necessarily elliptic. Complex powers of an elliptic pseudo-differential operator were defined by Seeley [5]. He constructed complex powers of a pseudo-differential operator $p(x, D_x)$ defined on a compact C^∞-manifold without boundary. Here we shall construct symbols for complex powers only by local calculation which works even for operators defined locally.

Recently Nagase-Shinkai [4] gave a concrete representation of complex powers of a pseudo-differential operator. They got the formula by using algebraic relation for the symbol of a pseudo-differential operator. But their method is not applicable to the case of systems, because they essentially used the commutativity of symbols.

We shall adopt the method of the Dunford integral for the symbol of the parametrix for $(p(x, D_x)-\zeta I)$. The relation between parametrices for $(p(x, D_x)-\zeta_1 I)$ and $(p(x, D_x)-\zeta_2 I)$, called the quasi-resolvent equation, plays an important role in place of the resolvent equation.

In the case of a single operator complex powers in the present paper coincide with those in [4]. We also note that complex powers of a parabolic system are asymptotically equal to operators with kernels whose supports lie in the half-space.

§1. Main theorem. Let $\lambda(\xi)$ be a fixed basic weight function, that is, a C^∞-function on R^n which have properties: $(1+|\xi|)^\rho \leqq \lambda(\xi) \leqq C_0(1+|\xi|)$ for some $\rho (0<\rho\leqq 1)$ and $|\partial_\xi^\alpha \lambda(\xi)| \leqq C_\alpha \lambda(\xi)^{1-|\alpha|}$ for any α (cf. [4]), where $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multi-index, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n}$. We denote by S_λ^m the set of all C^∞-symbols $p(x, \xi)$ on $R^n \times R^n$ satisfying, for any multi-index α, β, $|\partial_\xi^\alpha D_x^\beta p(x, \xi)| \leqq C_{\alpha,\beta}\lambda(\xi)^{m-|\alpha|}$ for some constant $C_{\alpha,\beta}$, and we define the pseudo-differential operator $p(x, D_x)$ of class S_λ^m by

$$p(x, D_x)u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi,$$

where $D_x^\alpha = (-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$ and $\hat{u}(\xi)=\mathcal{F}[u](\xi)$ denotes the Fourier transform of a rapidly decreasing function $u(x)$ defined by

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)dx.$$

Set $S_\lambda^{-\infty}=\bigcap_{-\infty<m<\infty} S_\lambda^m$. Then, we have the following

Theorem. *Let $p(x,D_x)=(p_{ij}(x,D_x))$ be an $N\times N$-matrix of pseudo-differential operators $p_{ij}(x,D_x)$ of class S_λ^m. We set the assumptions:*

A) *Let $\zeta_1(x,\xi),\cdots,\zeta_N(x,\xi)$ be eigenvalues of $p(x,\xi)$. Then, we have, for some $\delta_0>0$, $|\zeta_j(x,\xi)|\geqq\delta_0\lambda(\xi)^m$ for $x,\xi\in R^n$, $j=1,2,\cdots,N$.*

B) *There exists a constant C such that* $\operatorname{dis}(\zeta_j(x,\xi),(-\infty,0])\geqq C\lambda(\xi)^m$.

Then, we have a one parameter family $\{p_z(x,\xi)\}$ of symbols $p_z(x,\xi)$ of class $S_\lambda^{m\mathcal{R}ez}$ for complex numbers z, such that, for any z_1,z_2,

$$p_{z_1}(x,D_x)p_{z_2}(x,D_x)\equiv p_{z_1+z_2}(x,D_x),\ p_1(x,D_x)\equiv p(x,D_x),\ p_0(x,D_x)\equiv I \quad (\mathrm{mod}\ S_\lambda^{-\infty}).$$

Remark 1°. In the case when the assumptions A) and B) hold for large ξ we take a C_0^∞-function $\psi(\xi)$ such that $0\leqq\psi(\xi)\leqq 1$, $\psi(\xi)=1$ for $|\xi|\leqq\tau$ and $=0$ for $|\xi|\geqq 2\tau(\tau>0)$, and consider $p^{(\tau)}(x,\xi)=p(x,\xi)+\tau\psi(\xi)I$ for a large τ. Then $p^{(\tau)}(x,\xi)$ satisfies A) and B), so that complex powers for $p^{(\tau)}(x,D_x)$ in the theorem exist and become those for $P(x,D_x)$ since $p^{(\tau)}(x,D_x)\equiv p(x,D_x)$ (mod $S_\lambda^{-\infty}$).

2°. When $p(x,\xi)$ is defined in $\Omega\times R^n$ for an open set Ω of R^n, and satisfies A) and B) there, the theorem holds in the sense: $a(x)(p_{z_1}b(x)p_{z_2}-p_{z_1+z_2})a(x)\equiv 0$, $a(x)(p_1-p)a(x)\equiv 0$ and $a(x)(p_0-I)a(x)\equiv 0$ (mod $S_\lambda^{-\infty}$), where $a(x)$ and $b(x)$ are functions of class $C_0^\infty(\Omega)$ such that $b(x)=1$ in a neighborhood of the support of $a(x)$.

§2. Construction of parametrix. Using constants in the assumptions A) and B) we set

$$\Sigma_0=\{\zeta\,;\ \operatorname{dis}(\zeta,(-\infty,0])\leqq 1/2\min(\delta_0,C)\},$$
$$\Sigma_\xi=\{\zeta\,;\ \operatorname{dis}(\zeta,(-\infty,0])\leqq 1/2\min(\delta_0,C)\lambda(\xi)^m\}.$$

Proposition 2.1. *Let $p(x,\xi)$ satisfy the assumptions* A) *and* B). *Then, there exists a parametrix $r(\zeta\,;X,D_x)$ of $(p(x,D_x)-\zeta I)$ for any $\zeta\in\Sigma_0$ in the sense*

$r(\zeta\,;x,D_x)(p(x,D_x)-\zeta I)\equiv(p(x,D_x)-\zeta I)r(\zeta\,;x,D_x)\equiv I$ (mod $S_\lambda^{-\infty}$),
and $r(\zeta\,;x,\xi)$ has the analytic extension to Σ_ξ and has the form

$$r(\zeta\,;x,\xi)=\sum_{j=0}^{\infty}\varphi_j(\xi)q_j(\zeta\,;x,\xi)\quad in\quad \Sigma_\xi,$$

where $\varphi_j(\xi)$ are C^∞-functions, and $q_j(\zeta\,;x,\xi)$ satisfy

$$q_0(\zeta\,;x,\xi)=(p(x,\xi)-\zeta I)^{-1},$$

$$(2.1)\quad \sum_{j=1}^{N}\sum_{|\alpha|=N-j}\frac{1}{\alpha!}\partial_\xi^\alpha q_j(\zeta\,;x,\xi)D_x^\alpha(p(x,\xi)-\zeta I)=0\quad in\quad \Sigma_\xi, N\geqq 1.$$

Furthermore, $r(\zeta\,;x,\xi)$ satisfies

$$(2.2)\quad |(\partial_\xi^\alpha D_x^\beta r)(\lambda(\xi)^m\eta\,;x,\xi)|\leqq C_{\alpha,\beta}(|\eta|+1)^{-1}\lambda(\xi)^{-m-|\alpha|}\quad for\ any\ \eta\in\Sigma_0.$$

The proposition is the direct consequence of the following

Lemma 2.2. *$q_j(\zeta\,;x,\xi)$ in Proposition 2.1 satisfy*

$$(2.3)\quad \begin{cases} \text{i)} & |q_0(\zeta\,;x,\xi)|\leqq C_0(\lambda(\xi)^m+|\zeta|)^{-1}, \\ \text{ii)} & |\partial_\xi^\alpha D_x^\beta q_0(\zeta\,;x,\xi)|\leqq C_{\alpha,\beta}(\lambda(\xi)^m+|\zeta|)^{-2}\lambda(\xi)^{m-|\alpha|} \\ & \qquad \text{in } \Sigma_\xi\ (|\alpha+\beta|\geqq 1). \end{cases}$$

$$(2.4)\quad |\partial_\xi^\alpha D_x^\beta q_j(\zeta\,;x,\xi)|\leqq C_{\alpha,\beta,j}(\lambda(\xi)^m+|\zeta|)^{-2}\lambda(\xi)^{m-j-|\alpha|} \quad \text{in } \Sigma_\xi\ (j\geqq 1).$$

$$(2.5)\quad |(\partial_\xi^\alpha D_x^\beta q_j)(\lambda(\xi)^m\eta\,;x,\xi)|\leqq C'_{\alpha,\beta,j}\lambda(\xi)^{-m-j-|\alpha|}(|\eta|+1)^{-1} \quad \text{for } \eta\in\Sigma_0\ (j\geqq 0).$$

From (2.3) and (2.4) we have $|\partial_\xi^\alpha D_x^\beta q_j(\zeta\,;x,\xi)|\leqq C_{\alpha,\beta,j}\lambda(\xi)^{-m-j-|\alpha|}$ in Σ_ξ. Then as in Theorem 2.7 of [2], we can choose a sequence of functions $\varphi_j(\xi)\in C^\infty(R^n)$ such that $\varphi_0(\xi)\equiv 1$, $\varphi_j(\xi)=0$ near $\xi=0$, and $=1$ near $|\xi|=\infty$ for $j>0$, and

$$\begin{cases} r(\zeta\,;x,\xi)=\sum_{j=0}^{\infty}\varphi_j(\xi)q_j(\zeta\,;x,\xi)\in S_\lambda^{-m}, \\ r(\zeta\,;x,D_x)(p(x,D_x)-\zeta I)\equiv I \qquad (\text{mod } S_\lambda^{-\infty}) \quad \text{in} \quad \Sigma_0. \end{cases}$$

Furthermore, by (2.4) we can choose functions $\varphi_j(\xi)$ so that $r(\zeta\,;x,\xi)$ satisfies $|(\partial_\xi^\alpha D_x^\beta r)(\zeta\,;x,\xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{-|\alpha|}(\lambda(\xi)^m+|\zeta|)^{-1}$ in Σ_ξ. Then, we have (2.2).

Proof of Lemma 2.2. The inequality i) of (2.3) is obvious from the assumptions A) and B). To prove the others, we shall use the explicit form of $\partial_\xi^\alpha D_x^\beta q_0(\zeta\,;x,\xi)$ and $\partial_\xi^\alpha D_x^\beta q_j(\zeta\,;x,\xi)$. For $|\alpha|=1$ we have $\partial_\xi^\alpha q_0=-q_0\partial_\xi^\alpha p q_0$, $D_x^\alpha q_0=-q_0 D_x^\alpha p q_0$, so that we get

$$(2.6)\qquad \partial_\xi^\alpha D_x^\beta q_0=\Sigma C_{l,\beta^1,\dots,\beta^l}^{\alpha^1,\alpha^2,\dots,\alpha^l} q_0 p_{(\beta^1)}^{(\alpha^1)} q_0 p_{(\beta^2)}^{(\alpha^2)} q_0\cdots q_0 p_{(\beta^l)}^{(\alpha^l)} q_0,$$

where $p_{(\beta)}^{(\alpha)}(x,\xi)=\partial_\xi^\alpha D_x^\beta p(x,\xi)$, and the summation is taken under the condition: $1\leqq l\leqq|\alpha+\beta|$, $\alpha^1+\cdots+\alpha^l=\alpha$, $\beta^1+\cdots+\beta^l=\beta$. Hence we have ii) of (2.3). From (2.1) we can easily see that $\partial_\xi^\alpha D_x^\beta q_j$ also have the form (2.6) and we get (2.4). The inequality (2.5) is clear from (2.3) and (2.4).

§3. Quasi-resolvent equation. Proposition 3.1. *Let $r(\zeta_j\,;x,\xi)$, $j=1,2$, be the symbols of parametrices for $(p(x,D_x)-\zeta_j I)$. Then we have the quasi-resolvent equation in the sense*

$$(3.1)\quad \begin{aligned} & r(\zeta_1\,;x,D_x)r(\zeta_2\,;x,D_x) \\ & \quad =(\zeta_2-\zeta_1)^{-1}(r(\zeta_1\,;x,D_x)-r(\zeta_2\,;x,D_x))+R(\zeta_1,\zeta_2\,;x,D_x), \end{aligned}$$

where $R(\zeta_1,\zeta_2\,;x,\xi)\in S_\lambda^{-\infty}$, and satisfies for any α,β, and real s

$$(3.2)\quad |\partial_\xi^\alpha D_x^\beta R(\zeta_1,\zeta_2\,;x,\xi)|\leqq C_{\alpha,\beta,s}(|\zeta_1-\zeta_2|(|\zeta_1|+1)(|\zeta_2|+1))^{-1}\lambda(\xi)^s,$$

where $C_{\alpha,\beta,s}$ depends only on α,β and s.

Proof. By Proposition 2.1 we have, for some $K_1,K_2\in S_\lambda^{-\infty}$, $r(\zeta_1)(p-\zeta_1 I)=I+K_1(\zeta_1)$ and $(p-\zeta_2 I)r(\zeta_2)=I+K_2(\zeta_2)$. Considering $r(\zeta_1)(p-\zeta_1 I)r(\zeta_2)-r(\zeta_1)(p-\zeta_2 I)r(\zeta_2)$, we have $(\zeta_2-\zeta_1)r(\zeta_1)r(\zeta_2)=r(\zeta_2)-r(\zeta_1)+K_1(\zeta_1)r(\zeta_2)-r(\zeta_1)K_2(\zeta_2)$. Then, setting $R(\zeta_1,\zeta_2\,;x,D_x)=(\zeta_2-\zeta_1)^{-1}\times(K_1(\zeta_1)r(\zeta_2)-r(\zeta_1)K_2(\zeta_2))$, we get (3.1). Since $K_1,K_2\in S_\lambda^{-\infty}$. By the formula for the symbol of the product of pseudo-differential operators in [3], we have for any N

$$\sigma(r(\zeta_1)(p-\zeta_1 I))(x,\xi)$$
(3.3)
$$=\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\xi^\alpha r(\zeta_1;x,D_x)D_x^\alpha(p(x,\xi)-\zeta_1 I)+R_N(\zeta_1;x,\xi),$$

where

$$R_N(\zeta_1;x,\xi)=(2\pi)^{-n}\iint e^{-iy\cdot\mu}\langle y\rangle^{-n_0}\langle D_\mu\rangle^{n_0}$$
$$\times N\sum_{|\gamma|=N}\frac{\mu^\gamma}{\gamma!}\int_0^1(1-t)^{N-1}\partial_\xi^\gamma r(\zeta_1;x,\xi+t\mu)\cdot(p(x+y,\xi)-\zeta_1 I)dtdyd\mu$$

($\langle y\rangle^{-n_0}=(1+|y|^2)^{-n_0/2}$ and $\langle D_\mu\rangle^{n_0}=(1+|D_\mu|^2)^{n_0/2}$ for even $n_0\geqq n+1$).

We write $r(\zeta;x,\xi)=\sum_{j=0}^{N-1}q_j(\zeta;x,\xi)+\sum_{j=1}^{N-1}(\varphi_j(\xi)-1)q_j(\zeta;x,\xi)+\sum_{j=N}^{\infty}\varphi_j(\xi)q_j(\zeta;x,\xi)$. Then, using (2.1), we have

$$K_1(\zeta_1;x,\xi)=\sum_{|\alpha|<N}\sum_{1\leqq j\leqq N-1}\frac{1}{\alpha!}\partial_\xi^\alpha((1-\varphi_j)q_j(\zeta_1))D_x^\alpha(p-\zeta_1 I)$$
$$+\sum_{|\alpha|<N}\sum_{j=N}^{\infty}\frac{1}{\alpha!}\partial_\xi^\alpha((\varphi_j q_j(\zeta_1))D_x^\alpha(p-\zeta_1 I)$$
$$+N(2\pi)^{-n}\sum_{|\gamma|=N}\iint\frac{\mu^\gamma}{\gamma!}e^{-iy\cdot\mu}\langle y\rangle^{-n_0}\langle D_\mu\rangle^{n_0}\int_0^1\partial_\xi^\gamma r(\zeta_1;x,\xi+t\mu)$$
$$\times(p(x+y,\xi)-\zeta_1 I)dtdyd\mu\equiv I_1+I_2+I_3.$$

Using (2.4) and noting $(\varphi_j(\xi)-1)$ have compact supports for $j\geqq 1$, we have for any α',β' and real s

(3.4) $$|\partial_\xi^{\alpha'}D_x^{\beta'}I_1|\leqq C_{N,\alpha',\beta',s}\lambda(\xi)^{m-s}(\lambda(\xi)^m+|\zeta_1|)^{-1}.$$

Since $j\geqq N$ in I_2, we have (2.4)

(3.5) $$|\partial_\xi^{\alpha'}D_x^{\beta'}I_2|\leqq C_{N,\alpha',\beta'}\lambda(\xi)^{m-N-|\alpha'|}(\lambda(\xi)^m+|\zeta_1|)^{-1}.$$

If we estimate I_3 by the similar way to the estimate of the remainder term in the expansion formula for the symbol of the product of pseudo-differential operators in [3], then, using (2.3) and (2.4) we have

(3.6) $$|\partial_\xi^{\alpha'}D_x^{\beta'}I_3|\leqq C'_{N,\alpha',\beta'}\lambda(\xi)^{m-N-|\alpha'|}(\lambda(\xi)^m+|\zeta_1|)^{-1}.$$

Consequently, for any α,β and real s we have, from (3.3), (3.4)–(3.6),

$$|\partial_\xi^\alpha D_x^\beta K_1(\zeta_1;x,\xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{m-s}(\lambda(\xi)^m+|\zeta_1|)^{-1}$$

by fixing large N. We get the same estimate for $K_2(\zeta_2;x,\xi)$ and get (3.2).

§ 4. Proof of Theorem.

For $\mathscr{R}e\, z<0$ we define $p_z(x,\xi)$ by

$$p_z(x,\xi)=(2\pi i)^{-1}\int_\Gamma \zeta^z r(\zeta;x,\xi)d\zeta,$$

where Γ denotes a curve begining at $-\infty$, passing along $(-\infty,0]$, turning around the origin counterclockwise, and back to $-\infty$ along $(-\infty,0]$ (Γ can be chosen disjoint from the eigenvalues of $p(x,\xi)$ and dis$(\Gamma,(-\infty,0])=\delta_1>0$ for some δ_1). Then we have

$$\partial_\xi^\alpha D_x^\beta\int_\Gamma\zeta^z r(\zeta;x,\xi)d\zeta=\int_\Gamma\zeta^z(\partial_\xi^\alpha D_x^\beta r)(\zeta;x,\xi)d\zeta.$$

Since the integrand is analytic function of ζ in Σ_ε uniformly, we have, noting (2.2),

$$|\partial_\xi^\alpha D_x^\beta p_z(x,\xi)|$$
$$\leqq (2\pi)^{-1}\int_\Gamma |(\lambda(\xi)^m\eta)^z|\,|(\partial_\xi^\alpha D_x^\beta r)(\lambda(\xi)^m\eta\,;x,\xi)|\,\lambda(\xi)^m\,|d\eta|$$
$$\leqq (2\pi)^{-1}\int_\Gamma |\eta^z|(|\eta|+1)^{-1}|d\eta|\,\lambda(\xi)^{m\mathcal{R}_e z-|\alpha|}.$$

Hence, we get $p_z(z,\xi)\in S_\lambda^{m\mathcal{R}_e z}$.

Next we take two curves Γ_1,Γ_2 in Σ_0 like Γ such that $\mathrm{dis}(\Gamma_1,\Gamma_2)\geqq\delta_2>0$ and Γ_2 lies inside Γ_1. From the uniform analyticity of $r(\zeta\,;x,\xi)$ in ζ and (2.2) we have

$$p_{z_j}(x,\xi)=(2\pi i)^{-1}\int_{\Gamma_j}\zeta_j^{z_j}r(\zeta_j,x,\xi)d\zeta_j \qquad \text{for } \mathcal{R}_e\, z_j<0,\ j=1,2.$$

Consider $p_{z_1}(x,D_x)p_{z_2}(x,D_x)u(x)$ for a rapidly decreasing function $u(x)$. Then, by Fubini's theorem, we have

$$p_{z_1}(x,D_x)p_{z_2}(x,D_x)u(x)$$
$$=(2\pi i)^{-2}\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}r(\zeta_1\,;x,D_x)r(\zeta_2\,;x,D_x)u(x)d\zeta_1 d\zeta_2.$$

Now we apply Proposition 3.1 to $r(\zeta_1\,;x,D_x)r(\zeta_2,x,D_x)$. Then noting

$$\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}(\zeta_1-\zeta_2)^{-1}r(\zeta_2\,;x,D_x)u(x)d\zeta_2=0,$$

and by (3.2)

$$\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}R(\zeta_1,\zeta_2\,;x,D_x)d\zeta_1 d\zeta_2\in S_\lambda^{-\infty},$$

we have $p_{z_1}(x,D_x)p_{z_2}(x,D_x)u(x)=p_{z_1+z_2}(x,D_x)u(x)\ (\mathrm{mod}\, S_\lambda^{-\infty})$. Since $p_{-1}(x,\xi)=r(0\,;x,\xi)$, $p_{-1}(x,\xi)$ is the parametrix for $p(x,D_x)$. Hence, defining $p_{z+1}(x,D_x)$ by $p(x,D_x)p_z(x,D_x)$ the proof is completed.

§ 5. Example. Let $p(x,\xi)=(i\xi_n I-p_0(x,\tilde{\xi}))$, $\tilde{\xi}=(\xi_1,\cdots,\xi_{n-1})$, be a parabolic system of differential operators such that A) and B) hold for $\lambda(\xi)=(1+\xi_n^2+|\tilde{\xi}|^{2m})^{1/(2m)}$ (see [1], p. 239). Then $K_0(x,w)=(2\pi)^{-n}\times\int e^{iw\cdot\xi}p(x,\xi)^{-1}d\xi$ has the support in the half-space: $\{w\in R^n\,;w_n\geqq 0\}$. Noting $\partial_\xi^\alpha D_x^\beta p(x,\xi)$, $\alpha+\beta\neq 0$, are functions independent of ξ_n and using the form (2.1) and (2.6), we see that $p_z(x,D_x)$ are asymptotically equal to operators whose distribution kernels $K(x,w)$ have supports in the half-space in w and have, for any $\delta>0$, the estimate of the form

$$|K(x,w)|\leqq C_\delta(w_n)^{-n/m}\exp\{-c(|\tilde{w}|^m/w_n)^{1/(m-1)}\},\qquad w_n>\delta.$$

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