78. Uniqueness of the Plancherel Measure as an Invariant Measure over the Dual Objects of Compact Groups

By Nobuhiko TATSUUMA
Department of Mathematics, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

1. In the previous paper [1], we showed an invariancy of the Plancherel measure μ over the dual object $\hat{G}$ of unimodular locally compact groups G of type I under the Kronecker product operations. Especially, for the case that G is compact, the Corollary in [1] gives this invariancy as follows.

Proposition. *For any irreducible (so finite dimensional) unitary representation ω_0, and any function f in $L^1(G)\cap L^2(G)$,*

$$(\dim \omega_0)^{-1}\int_{\hat{G}} |||U_f(\omega_0\otimes\omega)|||^2\, d\mu(\omega)=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \qquad (1)$$

Here $|||U_f(\omega)|||$ is the Hilbert-Schmidt norm of the operator $U_f(\omega) \equiv\int_G f(g)U_g(\omega)dg$, corresponding to unitary representation $\omega =\{\mathfrak{H}(\omega), U_g(\omega)\}$ of G. (dg is the normalized Haar measure over G as $\int_G dg=1$.)

In this compact case, as is well known, the integral with respect to μ is just given by the summation with the weight "$\dim \omega$", the dimension of irreducible representation ω, over the discrete dual $\hat{G}$. That is

$$\int_{\hat{G}} \varphi(\omega)d\mu(\omega)=\sum_{\omega\in\hat{G}} \varphi(\omega)(\dim \omega). \qquad (2)$$

In the present paper, we shall show that the measure (i.e. weight function) satisfying such a invariancy is unique up to constant factor, that is, the following theorem.

Theorem. *Let $\nu(\omega)$ be a function over the dual $\hat{G}$ of a compact group G.*

If ν satisfies the following equation for any irreducible unitary representation ω_0 of G and for any function f in $L^1(G)\cap L^2(G)$,

$$(\dim \omega_0)^{-1}\sum_{\omega\in\hat{G}} |||U_f(\omega_0\otimes\omega)|||^2\nu(\omega)=\sum_{\omega\in\hat{G}} |||U_f(\omega)|||^2\nu(\omega), \qquad (3)$$

then there exists a constant c such that

$$\nu(\omega)=c\,(\dim \omega). \qquad (4)$$

Moreover, considering the Plancherel formula for G,

$$\int_G |f(g)|^2dg=\int_{\hat{G}} |||U_f(\omega)|||^2d\mu(\omega), \qquad (5)$$

we obtain a characterization of the Plancherel measure.

Corollary. *The Plancherel measure μ for compact group G is the unique measure over the dual object $\hat{G}$, which is invariant under the Kronecker product operations and satisfies*

$$\mu(\mathbf{1})=1 \tag{6}$$

Here **1** *shows the trivial representation of G.*

In fact, if we put $f(g)\equiv 1$ in the equation (5), (6) is deduced. The uniqueness of such a measure is direct result of the theorem.

2. Proof of the theorem. For compact group, the function of constant 1 is in $L^1(G)\cap L^2(G)$, then we can put $f\equiv 1$ in (3). Because of the orthogonality of matrices elements of irreducible representations, it is easy to see that $U_f\equiv U_1$ is zero except over the **1**-components (trivial representation). Therefore the square of the Hilbert-Schmidt norm of $U_f(\omega)$ is equal to the multiplicity $d(\omega)$ of **1**-components in ω. Especially,

$$|||U_f(\omega_0\otimes\omega)|||^2=d(\omega_0\otimes\omega) \tag{7}$$

But by the reason of the theory of finite dimensional irreducible representations,

$$\begin{aligned} d(\omega_0\otimes\omega)&=1, \quad \text{if} \quad \omega\sim\omega_0^*. \\ &=0, \quad \text{otherwise.} \end{aligned} \tag{8}$$

Here ω_0^* is the adjoint of ω_0 (cf. [2] p. 113).

Substitute the relations (7), (8) into (3). Then we get, for any irreducible representation ω_0,

$$(\dim\,\omega_0)^{-1}\nu(\omega_0)=\nu(\mathbf{1}). \tag{9}$$

This proves the theorem.

References

[1] N. Tatsuuma: Invariancy of Plancherel measure under the operation of Kronecker product. Proc. Japan Acad., **47**, 252–256 (1971).

[2] G. W. Mackey: Induced representations of locally compact groups. I. Ann. of Math., **55**, 101–139 (1952).