76. *The Theory of Nuclear Spaces Treated by the Method of Ranked Space. I*

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1. Introduction. In this paper we will show that the nuclear space in Gel'fand [2] can be considered as the limiting space of finite dimensional Euclidean space, when the limiting process is taken in the sense of ranked space given by K. Kunugi.

Following Gel'fand [2], the nuclear space Φ is a countably Hilbert space $\Phi=\bigcap_{i=1}^{\infty}\Phi_i$, in which for any m there is an n such that the mapping T_m^n, $m<n$, of the space Φ_n into the space Φ_m is nuclear, i.e., has the form

$$T_m^n\varphi=\sum_{k=1}^{\infty}\lambda_k(\varphi,\varphi_k)_n\psi_k, \qquad \varphi\in\Phi_n,$$

where $\{\varphi_k\}$ and $\{\psi_k\}$ are orthonormal systems of vectors in the space Φ_n and Φ_m respectively, $\lambda_k>0$ and $\sum_{k=1}^{\infty}\lambda_k$ converges.

§2. Definition of neighbourhoods. Let the mappings $T_{n_0}^{n_1}, T_{n_1}^{n_2}, \cdots, T_{n_{i-1}}^{n_i}, T_{n_i}^{n_{i+1}}, \cdots, (n_0=1<n_1<n_2<\cdots<n_{i-1}<n_i<n_{i+1}<\cdots)$ be nuclear operators in the nuclear space Φ. As shown in §1, we can write $T_{n_i}^{n_{i+1}}(i=0,1,2,\cdots)$ in the following form

$$T_{n_i}^{n_{i+1}}\varphi=\sum_{k=1}^{\infty}\lambda_{k,n_i,n_{i+1}}(\varphi,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}$$

where $\lambda_{k,n_i,n_{i+1}}>0$ and $\sum_{k=1}^{\infty}\lambda_{k,n_i,n_{i+1}}<\infty$. Now, we define

$$U_i(0,\varepsilon,m)=\left\{T_{n_{i-1}}^{n_i}\varphi:\varphi\in\Phi_{n_i}\cap\Phi\,\middle|\,\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(\varphi,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}<\varepsilon\right\}$$

as neighbourhoods of the origin of Φ and we call them neighbourhoods of rank i.

Lemma 1. *If we have* $m_i\leqq m_{i+1}$ and $(\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i})\varepsilon_{i+1}\leqq\varepsilon_i$, *we obtain*

$$U_i(0,\varepsilon_i,m_i)\supseteq U_{i+1}(0,\varepsilon_{i+1},m_{i+1}).$$

Proof. Suppose that $U_{i+1}(0,\varepsilon_{i+1},m_{i+1})\ni T_{n_i}^{n_{i+1}}\varphi$, $\varphi\in\Phi_{n_{i+1}}\cap\Phi$, then $\|\sum_{k=1}^{m_{i+1}}\lambda_{k,n_i,n_{i+1}}(\varphi,\varphi_{k,n_{i+1}})_{n_{i+1}}\varphi_{k,n_i}\|_{n_i}<\varepsilon_{i+1}$. Hence we obtain

$$\begin{aligned}
&\left\|\sum_{k=1}^{m_i}\lambda_{k,n_{i-1},n_i}(T_{n_i}^{n_{i+1}}\varphi,\varphi_{k,n_i})_{n_i}\varphi_{k,n_{i-1}}\right\|_{n_{i-1}}\\
&=\left\|\sum_{k=1}^{m_i}\lambda_{k,n_{i-1},n_i}\left(\sum_{h=1}^{\infty}\lambda_{h,n_i,n_{i+1}}(\varphi,\varphi_{h,n_{i+1}})_{n_{i+1}}\varphi_{h,n_i},\varphi_{k,n_i}\right)_{n_i}\varphi_{k,n_{i-i}}\right\|_{n_{i-1}}\\
&\leqq\left(\sum_{k=1}^{m_i}\lambda_{k,n_{i-1},n_i}\right)\left\|\sum_{h=1}^{m_{i+1}}\lambda_{h,n_i,n_{i+1}}(\varphi,\varphi_{h,n_{i+1}})_{n_{i+1}}\varphi_{h,n_i}\right\|_{n_i}\\
&<\left(\sum_{k=1}^{\infty}\lambda_{k,n_{i-1},n_i}\right)\varepsilon_{i+1}\leqq\varepsilon_i, \text{ then } T_{n_{i-1}}^{n_i}(T_{n_i}^{n_{i+1}}\varphi)\in U_i(0,\varepsilon_i,m_i).
\end{aligned}$$

Since we can identify $T^{n_i}_{n_{i-1}}(T^{n_{i+1}}_{n_i}\varphi)$ with $T^{n_{i+1}}_{n_i}\varphi$ in $\Phi_{n_{i-1}}$, we assert $U_i(0, \varepsilon_i, m_i) \supseteqq U_{i+1}(0, \varepsilon_{i+1}, m_{i+1})$.

Lemma 2. *If the following conditions*

(i) $0 < 2\left(\sum_{k=1}^{\infty} \lambda_{k,n_{i-1},n_i}\right)\varepsilon_{i+1} \leqq \varepsilon_i$

(ii) $m_i \leqq m_{i+1}$, $m_i \to \infty$,

are satisfied, we obtain

$$U_1(0, \varepsilon_1, m_1) \supseteqq U_2(0, \varepsilon_2, m_2) \supseteqq \cdots \supseteqq U_i(0, \varepsilon_i, m_i) \supseteqq \cdots$$

and $\bigcup_{i=1}^{\infty} U_i(0, \varepsilon_i, m_i) = 0$

Proof. Under the hypothesis, Lemma 1 leads to

$$U_i(0, \varepsilon_i, m_i) \supseteqq U_{i+1}(0, \varepsilon_{i+1}, m_{i+1}) \text{ for any } i.$$

Let us now verify the second part. To do this, it is necessary to show that for any $g \neq 0$ in Φ, there exists $U_i(0, \varepsilon_i, m_i)$ to which g does not belong.

Since $g \neq 0$, there exist some n_i and ε such that $\|g\|_{n_i} > \varepsilon > 0$. Since $\sum_{k=1}^{\infty} \lambda_{k,n_i,n_{i+1}}$ converges, we can take some m such that

$$\left\| \sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i} \leqq \left(\sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} \right) \|g\|_{n_{i+1}} < \frac{\varepsilon}{2}.$$

And we have

$$\|g\|^2_{n_i} = \left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|^2_{n_i} + \left\| \sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|^2_{n_i},$$

hence

$$\left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i} > \frac{\sqrt{3}}{2}\varepsilon.$$

Consequently, $U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m\right) \not\ni g$.

Let us here investigate the following three cases.

Case A. $m \leqq m_{i+1}$, $\quad \frac{\sqrt{3}}{2}\varepsilon \geqq \varepsilon_{i+1}$.

Since it is immediate that $U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m\right) \supseteqq U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m_{i+1}\right)$ $\supseteqq U_{i+1}(0, \varepsilon_{i+1}, m_{i+1})$, g does not belong to $U_{i+1}(0, \varepsilon_{i+1}, m_{i+1})$.

Case B. $m \leqq m_{i+1}$, $\quad \frac{\sqrt{3}}{2}\varepsilon < \varepsilon_{i+1}$.

For brevity, set $\left(\sum_{k=1}^{\infty} \lambda_{k,n_{i+h},n_{i+h+1}}\right) = A_{i+h}$, and Lemma 1 leads to the following series.

$$U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m\right) \supseteqq U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m_{i+1}\right) \supseteqq U_{i+2}\left(0, \frac{\sqrt{3}}{2}\Big/ A_i, m_{i+2}\right)$$
$$\supseteqq U_{i+3}\left(0, \frac{\sqrt{3}}{2}\varepsilon\Big/ A_i \cdot A_{i+1}, m_{i+3}\right)$$

$$\supseteqq \cdots \supseteqq U_{i+j+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}, m_{i+j+1}\right).$$

On the other hand, the hypotheses lead to the following series of inequalities,

$$2A_i\varepsilon_{i+2} \leqq \varepsilon_{i+1}$$
$$2A_{i+1}\varepsilon_{i+3} \leqq \varepsilon_{i+2}$$
$$\cdots\cdots\cdots\cdots$$
$$2A_{i+j-1}\varepsilon_{i+j+1} \leqq \varepsilon_{i+j}$$

and it follows from these that $\varepsilon_{i+j+1}(2^j \prod_{h=0}^{j-1} A_{i+h}) \leqq \varepsilon_{i+1}$.

We shall here take some integer j such that

$$\left(\frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}\right) > \left(\varepsilon_{i+1} \Big/ 2^j \prod_{h=0}^{j-1} A_{i+h}\right).$$

At once we have

$$\left(\frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}\right) > \varepsilon_{i+j+1}.$$

Hence we obtain

$$U_{i+j+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}, m_{i+j+1}\right) \supseteqq U_{i+j+1}(0, \varepsilon_{i+j+1}, m_{i+j+1}).$$

Thus we see that g is not contained in $U_{i+j+1}(0, \varepsilon_{i+j+1}, m_{i+j+1})$.

Case C. $m > m_{i+1}$.

In this case, we take some integer j such that $m < m_{i+j}$. In the similar way to the case B, we have

$$U_{i+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon, m\right) \supseteqq U_{i+j+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}, m\right)$$
$$\supseteqq U_{i+j+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}, m_{i+j+1}\right).$$

If $\left(\frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}\right) \geqq \varepsilon_{i+j+1}$, we have

$$U_{i+j+1}\left(0, \frac{\sqrt{3}}{2}\varepsilon \Big/ \prod_{h=0}^{j-1} A_{i+h}, m_{i+j+1}\right) \supseteqq U_{i+j+1}(0, \varepsilon_{i+j+1}, m_{i+j+1})$$

and we know that g does not belong to $U_{i+j+1}(0, \varepsilon_{i+j+1}, m_{i+j+1})$.

Otherwise, since we can choose some integer l such that

$$\left(\frac{\sqrt{3}}{2}\varepsilon \Big/ \left(\prod_{h=0}^{j-1} A_{i+h}\right)\left(\prod_{h=0}^{l-1} A_{i+j+h}\right)\right) > \left(\varepsilon_{i+j+1} \Big/ 2^l \prod_{h=0}^{l-1} A_{i+j+h}\right) \geqq \varepsilon_{i+j+l+1},$$

we see that $g \notin U_k(0, \varepsilon_k, m_k)$ for $k = i+j+l+1$.

Thus we assert that for all $g \neq 0$ there exists a $U_i(0, \varepsilon_i, m_i)$ to which g does not belong in either case.

Lemma 3. *If a sequence $\{g_n\}$ is bounded in countably Hilbert space, then the following two conditions are equivalent.*

(A) *In every Φ_{n_i}, there exists some integer N to each $\varepsilon > 0$ such that $\|g_n\|_{n_i} < \varepsilon$ for all $n \geqq N$.*

(B) *To each given $U_{i+1}(0, \varepsilon, m)$ there corresponds some integer N such that $g_n \in U_{i+1}(0, \varepsilon, m)$ for all $n \geqq N$.*

Proof. We shall prove the implications (A)$\Rightarrow$(B)$\Rightarrow$(A).

(A)$\Rightarrow$(B) By the definition of the nuclear space, we have

$$g_n = \sum_{k=1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g_n, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i}$$

and then the hypothesis leads to

$$\left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g_n, \varphi_{k,n_{i+1}}) \varphi_{k,n_i} \right\|_{n_i} \leqq \left\| \sum_{k=1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g_n, \varphi_{k,n_{i+1}}) \varphi_{k,n_i} \right\|_{n_i} < \varepsilon.$$

Hence g_n is contained in $U_{i+1}(0, \varepsilon, m)$ for all $n \geqq N$.

(B)$\Rightarrow$(A) If it is not true, there exists some Φ_{n_i} and a subsequence $\{g_{n_k}\}$ such that $\|g_{n_k}\|_{n_i} \geqq \varepsilon$ for some $\varepsilon > 0$.

Since the sequence $\{g_n\}$ is bounded in countably Hilbert space, there exist numbers C_i $(i=1, 2, \cdots)$ such that $\|g_n\|_{n_i} \leqq C_i$.

Then we can take some integer m such that

$$\left(\sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} \right) \|g_{n_k}\|_{n_{i+1}} \leqq \left(\sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} \right) C_{i+1} \leqq \frac{1}{2} \varepsilon,$$

because $\sum_{k=1}^{\infty} \lambda_{k,n_i,n_{i+1}}$ converges.

And then we see

$$\left(\sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} \right) \|g_{n_k}\|_{n_{i+1}} \leqq \frac{1}{2} \|g_{n_k}\|_{n_i}$$

On the other hand, we have

$$\|g_{n_k}\|_{n_i}^2 = \left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g_{n_k}, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i}^2 + \left\| \sum_{k=m+1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g_{n_k}, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i}^2.$$

Consequently we obtain

$$\left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g_{n_k}, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\| > \frac{1}{2} \varepsilon,$$

and then the subsequence $\{g_{n_k}\}$ is not contained in $U_{i+1}(0, 1/2\varepsilon, m)$.

This is a contradiction.

Lemma 4. *If a sequence $\{g_n\}$ is bounded in countably Hilbert space, then the following two conditions are equivalent.*

(A) *$\{g_n\}$ is a cauchy sequence in every Φ_{n_i}.*

(B) *To each given $U_{i+1}(0, \varepsilon, m)$ there corresponds some integer N such that the relations $n \geqq N$ and $m \geqq N$ imply $g_n - g_m \in U_{i+1}(0, \varepsilon, m)$.*

Proof (A)$\Rightarrow$(B). Since $\{g_n\}$ is a cauchy sequence in Φ_{n_i}, for any $\varepsilon > 0$, there exists some integer N such that the relations $n \geqq N$ and $m \geqq N$ imply $\|g_n - g_m\|_{n_i} < \varepsilon$.

Then we have

$$\left\| \sum_{k=1}^{m} \lambda_{k,n_i,n_{i+1}} (g_n - g_m, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i} \leqq \left\| \sum_{k=1}^{\infty} \lambda_{k,n_i,n_{i+1}} (g_n - g_m, \varphi_{k,n_{i+1}})_{n_{i+1}} \varphi_{k,n_i} \right\|_{n_i} = \|g_n - g_m\|_{n_i} < \varepsilon,$$

and hence $g_n - g_m \in U_{i+1}(0, \varepsilon, m)$.

(B)$\Rightarrow$(A) If it is not true, i.e., there exists some Φ_{n_i} such that $\{g_n\}$ is not a cauchy sequence in Φ_{n_i}, then to some $\varepsilon > 0$ there exists the subsequence $\{g_{n_k}\}$ such that $\| g_{n_k} - g_{n_{k+1}} \|_{n_i} > \varepsilon$.

On the other hand, since the sequence $\{g_{n_k} - g_{n_{k+1}}\}$ is bounded and satisfies the condition of Lemma 3, (B), and then Lemma 3, (A) show a contradiction.

References

[1] K. Kunugi: Sur la méthode des espaces rangés. I, II. Proc. Japan Acad., **42**, 318–322, 549–554 (1966).

[2] I. M. Gel'fand and N. Ya. Vilenkin: Generalized Functions, Vol. 4 (1964).