106. *An Operator-Valued Stochastic Integral*

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1. Introduction. In this paper we define a stochastic integral of the form

$$\int_b^a \xi(t,\omega)dw(t,\omega) \tag{1}$$

where $\xi(t,\omega)$ is a second order Hilbert space-valued random function and $w(t,\omega)$ is a Hilbert space-valued Brownian motion or Wiener process. The stochastic integral to be defined is operator-valued; in particular, it is a function from a probability space into the space of Schmidt class operators on a Hilbert space. Hilbert space-valued stochastic integrals of operator-valued functions have been studied by several authors (cf., Mandrekar and Salehi [7], and Vakhaniya and Kandelski [10]). We first introduce some definitions and concepts which will be used in the development of the integral.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space, and let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A mapping $x:\Omega\to\mathfrak{H}$ is said to be a *random element* in $\mathfrak{H}$, or an *$\mathfrak{H}$-valued random variable*, if for each $y\in\mathfrak{H}$, $\langle x(\omega), y\rangle$ is a real-valued random variable. Similarly, a mapping $L:\Omega\to\mathcal{B}(\mathfrak{H})$ (where $\mathcal{B}(\mathfrak{H})$ is the Banach algebra of endomorphisms of $\mathfrak{H}$) is said to be a *random operator* if, for every $x, y\in\mathfrak{H}$, $\langle L(\omega)x, y\rangle$ is a real-valued random variable.

Let x and y be two given elements in $\mathfrak{H}$. The tensor product of x and y, written $x\otimes y$, is an endomorphism in $\mathfrak{H}$ whose defining equation is $(x\otimes y)h=\langle h, y\rangle x$, $h\in\mathfrak{H}$. A simple consequence of this definition is $(x_1\otimes y_1)(x_2\otimes y_2)=\langle x_2, y_1\rangle(x_1\otimes y_2)$. We refer to Schattan [8] for a discussion of the operator $x\otimes y$ and its properties. Now let $x(\omega)$ and $y(\omega)$ be two $\mathfrak{H}$-valued random variables; and consider the tensor product $x(\omega)\otimes y(\omega)$. Falb [3] (cf. also [5]) has shown that the operator-valued function $x(\omega)\otimes y(\omega)$ is measurable; i.e., it is a random operator. Falb established the measurability of $x(\omega)\otimes y(\omega)$ using open sets; however, it follows easily from the definitions of a random operator and the tensor product operator.

An $\mathfrak{H}$-valued random function $\{w(t,\omega), t\in[a,b]\}$ is said to be a

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Brownian motion or *Wiener process* in $\mathfrak{H}$ if (i) $\mathcal{E}\{w(t,\omega)\}=0$ for all $t\in[a,b]$, (ii) the increments of $w(t,\omega)$ over disjoint intervals are independent, (iii) $w(t,\omega)$ is a.s. continuous as a function of t, (iv) $\mathcal{E}\{\|w(t,\omega)-w(s,\omega)\|_{\mathfrak{H}}^2\}=\mathcal{E}\{\langle w(t,\omega)-w(s,\omega), w(t,\omega)-w(s,\omega)\rangle\}=|t-s|$, and (v) $\mathcal{E}\{\langle w(t_2,\omega)-w(s_2,\omega), U(w(t_1,\omega)-w(s_1,\omega))\rangle\}=0$ for $s_1<t_1\le s_2<t_2$, and $U\in\mathcal{B}(\mathfrak{H})$.

A random function $\xi(t,\omega)\in\mathfrak{H}$ is said to be *nonanticipative* of the process $w(t,\omega)$ if, for $r,s,t\in[a,b]$, $r\le s\le t$, $\xi(r,\omega)$ and $w(t,\omega)-w(s,\omega)$ are independent. Let H denote the Hilbert space of the equivalence classes of second order random functions $\xi(t,\omega)$; that is for every $t\in[a,b]$, $\xi(t,\omega)$ is a second order random element in $\mathfrak{H}$. The norm in H is $\|\xi\|_H=\left(\int_a^b \mathcal{E}\{\|\xi(t,\omega)\|_{\mathfrak{H}}^2\}dt\right)^{1/2}$. We remark that the class of all random functions in H nonanticipative of $w(t,\omega)$ is a linear manifold; and we denote its closure by H_w. Also, the set of all simple random functions nonanticipative of $w(t,\omega)$ is dense in H_w.

Finally, we need the notion of an operator of Schmidt class (cf., Dunford and Schwartz [2]). An operator A on $\mathfrak{H}$ is said to be a *Schmidt class operator* if, for a complete orthonormal sequence $\{e_i\}$ in $\mathfrak{H}$, $\|A\|_\sigma^2=\sum_{i=1}^\infty\|Ae_i\|^2<\infty$. The collection $[\sigma c]$ of Schmidt class operators is a Hilbert space with inner product $(A\,|\,B)=\sum_{i=1}^\infty\langle Ae_i, Be_i\rangle$ and norm $\|\cdot\|_\sigma$, the so-called Schmidt norm.

2. Definition of the integral. Some properties. In defining the stochastic integral, and in the study of its properties, we restrict our attention to random functions $\xi(t,\omega)$ in H_w. We first define the integral for simple random functions, and then extend it to all random functions in H_w.

Let $\xi(t,\omega)$ be a simple random function; that is, if $a=t_0<t_1<\cdots<t_{n-1}<t_n=b$, then

$$\xi(t,\omega)=\begin{cases}\xi(t_i,\omega), & t\in[t_i,t_{i+1})\\ 0, & \text{otherwise.}\end{cases}$$

For a simple random function $\xi(t,\omega)$ the integral is defined by

$$\int_a^b \xi(t,\omega)dw(t,\omega)=\sum_{i=0}^{n-1}\xi(t_i,\omega)\otimes[w(t_{i+1},\omega)-w(t_i,\omega)]. \tag{2}$$

Let $I(\omega)$ denote the integral defined by (2). Clearly $I:\Omega\to[\sigma c]$; and $I(\omega)$ is a random operator of Schmidt class on $\mathfrak{H}$. Using elementary properties of the tensor product operator defined earlier, and properties of the processes $\xi(t,\omega)$ and $w(t,\omega)$, we obtain the following result for the integral defined by (2).

Lemma 1. (i) *For any two real numbers α_1 and α_2, and for simple random functions $\xi_1(t,\omega)$ and $\xi_2(t,\omega)$, we have*

$$\int_a^b [\alpha_1\xi_1(t,\omega)+\alpha_2\xi_2(t,\omega)]dw(t,\omega)$$

$$=\alpha_1\int_a^b \xi_1(t,\omega)dw(t,\omega)+\alpha_2\int_a^b \xi_2(t,\omega)dw(t,\omega).$$

(ii) *For a simple random function* $\xi(t,\omega)\in H_w$, $\xi\{I(\omega)\}=0$, *in the sense that* $\langle\mathcal{E}\{I(\omega)\}x,y\rangle=0$ *for every* $x, y\in\mathfrak{H}$.

(iii) *For a simple random function* $\xi(t,\omega)\in H_w$,

$$\mathcal{E}\left\{\left\|\int_a^b \xi(t,\omega)dw(t,\omega)\right\|_\sigma^2\right\}=\|\xi(t,\omega)\|_H^2$$

(iv) *For any* $U\in\mathcal{B}(\mathfrak{H})$ *and any simple random function* $\xi(t,\omega)\in H_w$,

$$\int_a^b U\xi(t,\omega)dw(t,\omega)=U\int_a^b \xi(t,\omega)dw(t,\omega).$$

(v) $\mathrm{Tr}\,[I(w)]=\sum_{i=0}^{n-1}\langle\xi(t_i,\omega), w(t_{i+1},\omega)-w(t_i,\omega)\rangle$, *and* $\mathcal{E}\{\mathrm{Tr}\,[I(\omega)]\}=0$.

Using (iii) of the above lemma, together with the following result, definition (2) can be extended to all $\xi(t,\omega)\in H_w$.

Lemma 2. *Let* $\{\xi_n(t,\omega)\}$ *be a Cauchy sequence of simple random functions in* H_w. *Then the corresponding integrals* $\{I_n\}$ *form a Cauchy sequence in* $L_2(\Omega,[\sigma c])$.

Let $\xi(t,\omega)\in H_w$. Then there exists a sequence $\xi_n(t,\omega)$ of simple random functions converging to $\xi(t,\omega)$ in H_w. Corresponding to $\{\xi_n(t,\omega)\}$, the integrals $I_n(\omega)=\int_a^b \xi_n(t,\omega)dw(t,\omega)$ form a Cauchy sequence in the Hilbert space $L_2(\Omega,[\sigma c])$. Thus, using the $L_2(\Omega,[\sigma c])$ convergence, the integral $\int_a^b \xi(t,\omega)dw(t,\omega)$, for all $\xi(t,\omega)\in H_w$, is defined by

$$\int_a^b \xi(t,\omega)dw(t,\omega)=\underset{n\to\infty}{\mathrm{l.i.m.}}\int_a^b \xi_n(t,\omega)dw(t,\omega). \tag{3}$$

Property (iii) of Lemma 1 defined an isometry from the simple random functions into $L_2(\Omega,[\sigma c])$; and since the simple random functions are dense in H_w, the mapping $\xi\to\int_a^b \xi dw$ extends by continuity to an isometry. Thus the definition of the stochastic integral can be formulated as follows:

Theorem 1. *There is a unique isometric operator from* H_w *into* $L_2(\Omega,[\sigma c])$, *denoted by*

$$\xi(t,\omega)\to\int_a^b \xi(t,\omega)dw(t,\omega).$$

The above result states that property (iii) of Lemma 1 holds for any $\xi(t,\omega)\in H_w$. By passing to the limit, properties (i) and (ii) of Lemma 1 also hold for any $\xi(t,\omega)\in H_w$.

Now, consider the operator-valued process $m(t,\omega)$ defined by

$$m(t,\omega)=\int_a^t \xi(t,\omega)dw(t,\omega),\qquad t\geq a. \tag{4}$$

We state the following result, which is an analogue of a well-known property of the Itô-Doob integral (cf., Doob [1], p. 444).

Theorem 2. *If $\xi(t,\omega)\in H_w$, then the process $m(t,\omega)$ defined by (4) is an operator-valued martingale.*

3. The covariance operator of the integral. Consider the measurable space $(\mathfrak{H},\mathscr{B})$ where $\mathfrak{H}$ is a real separable Hilbert space and $\mathscr{B}$ is the σ-algebra of Borel subsets of $\mathfrak{H}$. Let $x(\omega)$ denote a $\mathfrak{H}$-valued random variable; and let ν_x denote the probability measure on $\mathfrak{H}$ induced by μ and x, that is $\nu_x=\mu\circ x^{-1}$, or $\nu_x(B)=\mu(x^{-1}(B))$ for all $B\in\mathscr{B}$. Let $M(\mathfrak{H})$ denote the space of all probability measures on $\mathfrak{H}$; and let $\nu\in M(\mathfrak{H})$ be such that $\mathcal{E}_\nu\{\|x\|^2\}=\int\|x\|^2\,d\nu<\infty$. Then the *covariance operator* S of defined by the equation

$$\langle Sg,g\rangle=\int\langle f,g\rangle^2 d\nu(f) \tag{5}$$

(cf. Grenander [4], Chap. 6).

As a random element in $[\sigma c]$, the integral $I(\omega)$ induces a probability measure ν_I on the measurable space $([\sigma c],\mathscr{F})$, where $\mathscr{F}$ is the σ-algebra of Borel subsets of $[\sigma c]$; and $\nu_I=\mu\circ I^{-1}$. Now, if $\nu_I\in M([\sigma c])$ is such that $\int\|x\|^2 d\nu_I<\infty$; then it follows from (5) that the covariance operator S_I of the integral $I(\omega)$ is defined by

$$\langle S_I x,x\rangle=\int\langle y,x\rangle^2 d\nu_I(y) \tag{6}$$

The Hilbert space $L_2(\Omega,[\sigma c])$ is the tensor product of $L_2(\Omega)$ and $[\sigma c]$; that is $L_2(\Omega,[\sigma c])=L_2(\Omega)\hat{\otimes}[\sigma c]$ (cf., Umegaki and Bharucha-Reid [9]). Using tensor product methods, the authors [6] have obtained several representation theorems for covariance operators, which when applied to S_I give the following results.

Theorem 3. *The covariance operator S_I of the stochastic integral $I(\omega)$ admits the representation*

$$(S_I A\,|\,B)=\int_\Omega \mathrm{Tr}\,[(I(\omega)\otimes I(\omega))(A\otimes B)]d\mu,$$

where $A,B\in[\sigma c]$.

Theorem 4. *If $I(\omega)\in L_2(\Omega)\otimes[\sigma c]$ (the algebraic tensor product of $L_2(\Omega)$ and $[\sigma c]$), then S_I admits the representation*

$$(S_I A\,|\,A)=\sum_{i=1}^m\sum_{j=1}^n (x_i(\omega),y_i(\omega))(A_i\,|\,A)(A\,|\,A_j),$$

where $A\in[\sigma c]$, $(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$, and $I(\omega)=\sum_{i=1}^m x_i(\omega)\otimes A_i=\sum_{i=1}^m x_i(\omega)A_i$, with $x_i\in L_2(\Omega)$, $A_i\in[\sigma c]$, $l\leq i\leq m$.

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