105. A Theorem Equivalent to the Brouwer Fixed Point Theorem

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§0. Introduction.

In this note we shall give a theorem which is equivalent to the Brouwer fixed point theorem. Such a theorem, we shall call here Theorem A, can be applied to the foundation of analysis concerning several independent variables ([1] Lemma F).

Notations used here are the same as those in [1]. Let K be the n-dimensional closed unit ball, and K_δ be the closed ball of radius δ with center $\mathbf{0}$. Further let $(S)_{-\delta}$ be the maximal closed set whose δ-neighborhood is contained in the set S. The symbol $\|\cdot\|$ denotes the ordinary euclidean norm. The Brouwer fixed point theorem for a continuous mapping on K into itself is referred to as Theorem B.

Theorem A. *Let $f(x)$ be a continuous mapping defined on K into R^n of the form*

$$f(x)=Lx+N(x),$$

where L is non-degenerated affine mapping and $\|N(x)\|\leqq\delta$.
Then

$$f(K)\supset(LK)_{-\delta}.$$

For sufficiently small δ the set $(LK)_{-\delta}$ is not empty, and therefore such a continuous $f(x)$ in Theorem A may be considered as having the dimension-preserving property in some sense. Translating variables, C^1-mapping with non-vanishing Jacobian belongs to this class in local and Theorem A furnishes a lower bound of the extent of range $f(Q)$ for a small vicinity Q.

Theorem A increases in generality by certain modifications, however, we shall be interested in the fact that Theorem A which may be seen intuitively is equivalent to the Brouwer fixed point theorem.

§1. Theorem B implies Theorem A.

Proof. Let y be arbitrarily chosen from $(LK)_{-\delta}$ and fixed. Consider the mapping $x\rightarrow L^{-1}(y-N(x))$. Since $y-N(x)$ belongs to LK, this mapping is continuous on K into itself. Therefore by Theorem B there exists a fixed point $x(\in K)$ such that $L^{-1}(y-N(x))=x$ i.e. $y=Lx+N(x)$.

q.e.d.

§2. Theorem A implies TheoremB.

Proof. Suppose there exists a continuous mapping $f(x)$ on K into itself with no fixed point. Then there exists a continuous mapping

$f_1(x)$ such that $\|f_1(x)\|=1$ for all $x(\in K)$ and every boundary point remains fixed. Such a mapping $f_1(x)$ is obtained by solving $\lambda(\geqq 0)$ from the following quadratic equation $\|(1+\lambda)x-\lambda f(x)\|^2=1$.

Consider the mapping

$$F(t, x)=(1-t)x+tf_1(x)$$

for real parameter $t(0<t<1)$, where for a fixed t, $(1-t)x$ is the linear term Lx and $tf_1(x)$ is the non-linear term $N(x)$ respectively in Theorem A.

Evidently $\|tf_1(x)\|=t$, and we get

$$F(t, K)\supset((1-t)K)_{-t}=K_{1-2t}.$$

If $0<t<1/2$, then K_{1-2t} is not empty and $F(t, K)\ni \mathbf{0}$. By tending t to $\uparrow 1/2$, we get $F(1/2, K)\ni \mathbf{0}$ which means the existence of a point p such that $(1/2)p+(1/2)f_1(p)=\mathbf{0}$ i.e. $p=-f_1(p)$.

This last equation is the contradiction which says that p is boundary point and at the same time is removed by f_1. q.e.d.

Reference

[1] Z. Koshiba: An alternative method of foundation in the infinitesimal analysis of several independent variables. Journal of Shinshû Univ., Faculty of Science, **5**(2), 109–121 (1970).