104. *A Remark on the Concept of Channels. III*

An Algebraic Theory of Extended Toeplitz Operators

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In the previous notes [1], a few elementary properties of generalized channels are discussed. In the present note, some problems on extended Toeplitz operators will be studied as a kind of generalized channels.

1. In the classical theory of Toeplitz operators, a *Laurent operator* l_ϕ on L^2 is defined by the multiplication by an essentially bounded function ϕ with functions of $L^2(\varphi \in L^2 \to \phi\varphi \in L^2)$ where L^2 is the Hilbert space of all square integrable functions defined on the unit circle with the normalized Lebesgue measure. A *Toeplitz operator* t_ϕ is defined by

(1) $$t_\phi = pl_\phi \,|\, H^2,$$

where H^2 is the subspace of L^2 consisting those functions whose Fourier coefficients vanish on negative integers and where p is the projection belonging to H^2.

An abstraction of the above situation is recently given by Devinatz and Shinbrot [2]: An abstract Hilbert space H plays the role of L^2, and H^2 is replaced by an arbitrary (closed) subspace M. Every element a of $B(H)$, the algebra of all (bounded linear) operators, defines a *general Wiener-Hopf operator*

(1′) $$t_p(a) = pa \,|\, M,$$

where p is the projection belonging to M.

An another moderate abstraction is given by Douglas and Pearcy [4]. Every element of a maximally abelian von Neumann algebra A plays the role of Laurent operator. If each vector of M is separating in the sense of Dixmier [3] for A, M is called a *weak Riesz space*. If M and $M^\perp$ are weak Riesz subspaces for A, then M is called a *Riesz subspace*. A *Riesz system* is the triple (H, A, M). Every element $a \in A$ is called a *generalized Laurent operator* (simply (GL) *operator*) and $t_p(a)$ a *generalized Toeplitz operator* (simply (GT) *operator*).

2. Assume that A is a von Neumann algebra acting on H. Then the both cases are unified: $A = B(H)$ for the case of Devinatz-Shinbrot and A is maximally abelian for the case of Douglas-Pearcy. In the below, instead of $t_p(a)$, the following notation will be used:

(1″) $$a_p = pa \,|\, M.$$

The following proposition is easily checked:

I. *The mapping $a \to a_p$ from A into $B(pH)$ satisfies*

(2) $$(\alpha a+\beta b)_p=\alpha a_p+\beta b_p,$$

(3) $$(a_p)^*=(a^*)_p,$$

(4) $$a_p \geqq 0 \quad \text{if} \quad a \geqq 0,$$

(5) $$\|a_p\| \leqq \|a\|.$$

Since the mapping is normal in the sense of Dixmier [3] and since the image 1_p of the identity acts as the identity on $M=pH$, the requirements of generalized channels in [1] are satisfied. Hence *I* implies

II. *The mapping is a generalized channel.*

In the terminology of [1], $B(M)$ is the input and A is the output of the generalized channel. By II and [1; II, § 3], one has

III. *The closed numerical range $\bar{W}$ is contracted by the mapping: $\bar{W}(a) \supset \bar{W}(a_p)$. Especially, $\sigma(a_p) \subset \bar{W}(a)$, that is, the spectrum of the image is contained in the closed numerical range.*

In general, it is not decidable without further restriction that $\sigma(a_p)$ contains or is contained in $\sigma(a)$. For example, if

$$a=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad p=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $\sigma(a_p)=\{1\}$ whereas $\sigma(a)=\{0,2\}$.

3. Here a condition will be discussed which implies the so-called *Hartman-Wintner spectral inclusion theorem*:

(6) $$\sigma(a) \subset \sigma(a_p).$$

The following condition is essentially due to Devinatz and Shinbrot [2]:

(DS) *There is a set U of unitary operators in the commutator A' such that UM is dense in H.*

IV. *If the condition* (DS) *is satisfied, then the Hartman-Wintner Theorem holds. Moreover, the norm is preserved:*

(7) $$\|a_p\|=\|a\|.$$

For any $\xi \in M$, if a_p is invertible, then there is $\delta>0$ such as $\delta\|\xi\| \leqq \|a_p \xi\|$. Hence

$$\begin{aligned} \delta\|\xi\| \leqq \|a_p\xi\| \leqq \|a\xi\| = \|u^*ua\xi\| = \|ua\xi\| \\ = \|au\xi\| \end{aligned}$$

for every $u \in U \subset A'$. By (DS), UM is dense in H, so that a is $1:1$ on H and has closed range. Whereas the same is true for a^*; hence $aH=H$ and so a is invertible. This proves

V. *Under* (DS), *a is invertible if a_p is invertible.*

Hence (6) follows, which proves the first half.

For the second half,

$$\begin{aligned} \|a_p\| &= \sup\{|(a_p\xi \mid \eta)|;\ \xi, \eta \in M, \|\xi\|=\|\eta\|=1\} \\ &= \sup\{|(au\xi \mid u\eta)|;\ \xi, \eta \in M, \|\xi\|=\|\eta\|=1, u \in U\}=\|a\|. \end{aligned}$$

4. Let K be a generalized channel with the output A and the input B. Being taken into consideration of the previous section, K will be called a *Hartman-Wintner channel* if K satisfies

$$\sigma(Ka)\supset\sigma(a), \qquad \|a\|=\|Ka\|. \tag{8}$$

The definition implies at once

VI. *In a Hartman-Wintner channel,*

$$\sigma(a)\subset\sigma(Ka)\subset\bar{W}(Ka)\subset\bar{W}(a). \tag{9}$$

Let $r(a)$ be the spectral radius of a and $w(a)$ the numerical radius of a, then VI implies

VII. *In a Hartman-Wintner channel,*

$$r(a)\leqq r(Ka)\leqq w(Ka)\leqq w(a)\leqq\|a\|=\|Ka\|. \tag{10}$$

If co S denotes the convex hull of S, then (9) implies

$$\text{co }\sigma(a)\subset\text{co }\sigma(Ka)\subset\bar{W}(Ka)\subset\bar{W}(a),$$

which implies (10) (cf. [7]).

VIII. *Through a Hartman-Wintner channel, the closed numerical range of a convexoid is preserved:* $\bar{W}(Ka)=\bar{W}(a)$ *if* $\bar{W}(a)=\text{co }\sigma(a)$.

A similar observation on (10) has

IX. *A Hartman-Wintner channel preserves being spectraloid (resp. convexoid, normaloid).*

By [7], an operator a is a spectraloid if and only if $r(a)=w(a)$; hence (10) implies $r(Ka)=w(Ka)$, that is, Ka is a spectraloid.

Similarly, a is a normaloid if and only if $\|a\|=r(a)$; hence (10) implies $r(Ka)=\|Ka\|$, and so Ka is a normaloid.

The case for convexoids follows from VIII.

The following proposition is obvious by the definition:

X. *If K is a Hartman-Wintner channel, then Ka is not quasinilpotent if a is not quasinilpotent. Consequently, if A is abelian then KA contains no nonzero quasinilpotent element.*

It is remarked that IX is not a usual property of generalized channels. The reduction of a normaloid is not a normaloid, cf. [5].

5. Let A be a von Neumann algebra acting on H. If the mapping $a\to a_p$ for a fixed projection p (not necessarily belonging to A) is a Hartman-Wintner channel, then A is called a *Hartman-Wintner algebra* upon $M=pH$, and M is called a *Hartman-Wintner subspace* for A.

XI. *A von Neumann subalgebra B of a Hartman-Wintner algebra upon M is a Hartman-Wintner algebra upon M.*

For a fixed subspace M (or the projection p belonging to M), an operator a is called an *extended Laurent operator* (shortly (EL) *operator*) if a (and 1) generates a Hartman-Wintner algebra upon M, and a_p is called an *extended Toeplitz operator* (shortly (ET) *operator*). By the definition VIII and IX imply

XII. *The closed numerical range of a convexoid* (EL) *operator is* *identical with that of its* (ET) *operator*: $\bar{W}(a)=\bar{W}(a_p)$.

XIII. *If an* (EL) *operator is a spectraloid* (*resp. convexoid, normaloid*) *then its* (ET) *operator is a spectraloid* (*resp. convexoid, normaloid*) *too.*

Furthermore, (9) and (10) imply

$$\sigma(a)\subset\sigma(a_p)\subset\bar{W}(a_p)\subset\bar{W}(a), \tag{11}$$

$$r(a)\leqq r(a_p)\leqq w(a_p)\leqq w(a)\leqq\|a_p\|=\|a\|. \tag{12}$$

6. Suppose that A *is a Hartman-Wintner algebra upon* $M=pH$. An element $a\in A$ is *analytic* if $aM\subset M$ or $ap=pap$. If a is analytic, then a_p is called *analytic*. Let A_0 be the set of all analytic (ET) operators defined by A.

XIV. *If* $a_p\in A_0$, *then*

$$(ba)_p=b_pa_p \qquad (b\in A). \tag{13}$$

Since a is analytic, the definition implies

$$b_pa_p=(pbp)(pap)=pbpap=pbap=(ba)_p.$$

This implies also

XV. A_0 *is an algebra.*

By means of XV, A_0 will be called the *analytic algebra* of A. On A_0, the following version of the F. and M. Riesz Theorem is introduced:

(FM) *The support of any nonzero element of* A_0 *is* 1.

The first consequence of (FM) is

XVI. A_0 *contains no nonzero-zero-divisor.*

If $(ab)_p=0$ for a, $b\in A_0$ then $abp=0$ by the analyticity. If $b\neq 0$, then ran $bp\neq 0$ where ran c denotes the range of c; hence $\ker a\neq 0$ which contradicts with (FM) if $a\neq 0$ where $\ker d$ denotes the kernel of d. If $a\neq 0$, then (FM) implies ran $bp=0$, so that $b=0$ by (FM).

XVII. A_0 *contains no non-trivial idempotent.*

If $q\in A_0$ and $q^2=q$, then $q(q-1)=0$; hence by XVI either $q=0$ or $q=1$.

XVIII. *If* a_p *is analytic and invertible in* A_0, *then* a^{-1} *is also analytic and*

$$(a_p)^{-1}=(a^{-1})_p. \tag{14}$$

In a Hartman-Wintner algebra, the invertibility of a_p implies the existence of a^{-1}, and $1_p=(a^{-1}a)_p=(a^{-1})_pa_p$ by (13). Therefore $(a^{-1})_p$ is a left inverse of an invertible element a_p, and so (14) is proved.

Now, (14) implies

$$apa^{-1}p=papa^{-1}p=(a)_p(a^{-1})_p=p,$$

so that $pa^{-1}p=a^{-1}p$ and a^{-1} is analytic as desired.

It is deducible from XVIII that the spectra of a_p in A_0 and as operator coincide. Basing on this fact and assuming that A_0 is abelian, one enables to tail the proof of a theorem of Douglas and Pearcy [4]: The spectrum of an analytic Toeplitz operator is connected. The Gelfand representation, a theorem of Silov and XVII imply that the spectrum of A_0 is connected. Every $a_p \in A_0$ continuously maps the connected compact space onto $\sigma(a_p)$, so that $\sigma(a_p)$ is connected.

7. In the theory of general Wiener-Hopf operators, one of main problems is to determine a condition which insures the invertibility of a_p by that of a. Devinatz and Shinbrot [2] show that the strict positivity of the real part of a is sufficient, where c is *strictly positive* if there is $\delta > 0$ such as $c \geqq \delta > 0$. The following formal extension is possible:

XIX. *If zero is excluded by the closed numerical range of an operator a, then the Wiener-Hopf operator a_p is invertible for any projection p.*

By III, $\bar{W}(a_p) \subset \bar{W}(a)$; hence $0 \notin \bar{W}(a_p)$ by the hypothesis. Since $\sigma(a_p) \subset \bar{W}(a_p)$, $0 \notin \sigma(a_p)$ or a_p is invertible.

If a has the strictly positive real part, then 0 is not in $\bar{W}(a)$; hence XIX implies the theorem of Devinatz and Shinbrot. However, the implication is not proper. Berberian points out, $0 \notin \bar{W}(a)$ implies that the unitary part of the polar decomposition of a is "cramped"; hence a suitable rotation carries a into an operator with the strictly positive real part, cf. [8] for a proof and also cf. [6].

8. Basing on an idea of Poussin, Devinatz and Shinbrot [2] give a decomposition theorem: If a is invertible, then there are a unitary u and an invertible operator b such that $a = ub$ and b maps M onto itself. H. Choda gives the following generalization in his seminar talk:

XX. *If a and p belong to a von Neumann algebra A and a is invertible. Then there are a unitary u and an invertible b in A such that $a = ub$ and b maps M onto itself.*

Let $N = \operatorname{ran} ap$ and q be the projection belonging to N. Then

$$N = \operatorname{ran} ap = \operatorname{supp} pa^* \sim \operatorname{supp} ap = M,$$

where supp c denotes the support of c. Hence there is a partial isometry $v \in A$ such that

$$q = v^*v, \qquad p = vv^*.$$

By the definition, one has

$$\begin{aligned} N^\perp &= \ker pa^* = \{\xi \,;\, pa^*\xi = 0\} \\ &= \{\xi \,;\, a^*\xi \in (pH)^\perp\} \\ &= \{\xi \,;\, a^*\xi = p^\perp \eta \text{ for some } \eta \in H\} \\ &= \{\xi \,;\, \xi = a^{*-1} p^\perp \eta \text{ for some } \eta \in H\} \\ &= \operatorname{ran} a^{*-1} p^\perp. \end{aligned}$$

On the other hand, one has

$$N^\perp = \operatorname{supp} p^\perp a^{-1} \sim \operatorname{supp} a^{*-1} p^\perp = M^\perp\,;$$

hence there is a partial isometry $w \in A$ such that

$$q^\perp = w^*w, \qquad p^\perp = ww^*.$$

If $u = v + w$, then $u \in A$ is unitary and

$$uapH = u \text{ ran } ap = uqH = pH.$$

If $b = ua$, then b maps $M = pH$ onto M, and b is invertible since u and a are invertible, which completes the proof of XX.

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