103. *On Some Examples of Non-normal Operators*

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

1. Introduction. Following after the terminology of Halmos [4], consider a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$. As usual, we shall call

$$W(T)=\{(Tx|x);\ \|x\|=1\}$$

the *numerical range* of T and

$$r(T)=\sup\{|\lambda|;\ \lambda\in\sigma(T)\}$$

the *spectral radius* of T, where $\sigma(T)$ is the spectrum of T. An operator T is called *normaloid* if $\|T\|=r(T)$ and *convexoid* if $\bar{W}(T)=\mathrm{co}\,\sigma(T)$ where $\bar{W}(T)$ is the closure of $W(T)$ and $\mathrm{co}\,S$ is the convex hull of S. We shall also say that an operator T satisfies the *growth condition* (G_1) if

$$\|(T-\lambda)^{-1}\|\leqq\frac{1}{\mathrm{dist}\,(\lambda,\sigma(T))}$$

for any $\lambda\notin\sigma(T)$. An operator satisfying the condition (G_1) is a convexoid.

In a recent paper [7], Luecke proves the following theorem which gives a method of construction of operators satisfying the condition (G_1):

Theorem A (Luecke). *If A is an operator acting on a Hilbert space $\mathfrak{H}$, then there is an operator B acting on a Hilbert space $\mathfrak{K}$ such that their direct sum $T=A\oplus B$ acting on $\mathfrak{H}\oplus\mathfrak{K}$ satisfies the condition* (G_1).

In his proof, the desired B satisfies the normality and $\bar{W}(A)=\sigma(B)$. Using Theorem A, he can prove that there is an operator satisfying the condition (G_1) which is not a normaloid.

Inspired by Luecke's work and a seminar talk of R. Nakamoto (Theorem 5 in the below), we shall adapt the method to construct another examples of operators in § 2 and apply them to study for a few relations between classes of non-normal operators in § 3.

For our purpose, we shall introduce two classes of operators which are systematically discussed by Hildebrandt [5] without their names:

Definition B. An operator T is called a *numeroid* (resp. *spectroid*) if the closed numerical range $\bar{W}(T)$ (resp. the spectrum $\sigma(T)$) is a spectral set for T in the sense of von Neumann [8].

The author wishes to thank Prof. Luecke who gives an opportunity for the author to read [7] before publication.

2. Construction. We begin with the following simple case:

Theorem 1. *If A is an operator, then there is an operator B such that $T=A\oplus B$ is a normaloid.*

Proof. Take a normaloid B such as $\|B\|\geqq\|A\|$. Then

$$r(T)=\max\{r(A), r(B)\}=r(B)=\|B\|=\|T\|,$$

and T is a normaloid.

Theorem 2. *If A is an operator and B is a convexoid such as $\bar{W}(A)\subset\bar{W}(B)$, then $T=A\oplus B$ is a convexoid.*

Proof. Since $\bar{W}(B)=\mathrm{co}\,\sigma(B)$, we have

$$\begin{aligned}\bar{W}(T)&=\mathrm{co}\,\{\bar{W}(A)\cup\bar{W}(B)\}=\mathrm{co}\,\bar{W}(B)=\mathrm{co}\,\sigma(B)\\&=\mathrm{co}\,\{\sigma(A)\cup\sigma(B)\}=\mathrm{co}\,\sigma(T),\end{aligned}$$

so that T is a convexoid.

The following two theorems may justify our naming postfix "oid" in affinity with the previous two theorems.

Theorem 3. *If A is an operator which has S as a spectral set, and B is an operator which is a numeroid such as $S\subset\bar{W}(B)$, then $T=A\oplus B$ is a numeroid.*

Proof. If f is a rational function which has poles outside of $\bar{W}(B)$ and $\|f\|\leqq 1$ where $\|f\|=\sup\{|f(\lambda)|;\lambda\in\bar{W}(B)\}$, then we have

$$\begin{aligned}\|f(T)\|&=\|f(A\oplus B)\|=\|f(A)\oplus f(B)\|\\&\leqq\max\{\|f(A)\|,\|f(B)\|\}\leqq\|f\|\leqq 1\end{aligned}$$

since $\bar{W}(B)$ is a spectral set for A by a theorem of von Neumann [8], so that T is a numeroid.

Theorem 4. *If A has S as a spectral set and B is a spectroid with $S\subset\sigma(B)$, then $T=A\oplus B$ is a spectroid.*

Proof. One needs to replace $\sigma(B)$ instead of $\bar{W}(B)$ in Theorem 3, so that we shall omit the details.

Remark. By the above theorems, we can easily conclude that the class of all spectroids (resp. numeroids, convexoids, normaloids) is not invariant under the reduction.

3. Applications. Clancey [2] proves that a hyponormal operator needs not a spectroid. In the converse direction, we have

Theorem 5 (Nakamoto). *There is a spectroid which is not hyponormal.*

Proof. We wish to construct a spectroid using Theorem 4. Put

$$A=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

then we have

$$\sigma(A)=\{0\},\qquad W(A)=\frac{1}{2}D,\qquad \|A\|=1,$$

where D is the unit disc. Let B be the unilateral shift. Then we have by [4; Problem 67]

$$\sigma(B)=D, \quad W(B)=D, \quad \|B\|=1.$$

Hence B is a spectroid by a theorem of von Neumann. Put $T=A\oplus B$. Then by Theorem 4 T is a spectroid with

$$\sigma(T)=D, \quad W(T)=D, \quad \|T\|=1.$$

Whereas T is not hyponormal since the reduction of T on the first space $\mathfrak{H}$ (that is A itself) is a non-zero quasi-nilpotent and since the reduction of a hyponormal operator is also hyponormal (there is no non-zero hyponormal quasi-nilpotent operator [4; Problem 162]).

Corollary 1. *There is a numeroid which is not hyponormal.*

Corollary 2. *There is a spectroid which is not paranormal.*

Since the paranormality introduced by Istrăţescu [6] and named by Furuta [3] is invariant under the reduction, the same reasoning for Theorem 5 implies our conclusion.

By a theorem of Luecke [7], there is an operator satisfying the growth condition (G_1) which is not a normaloid; hence the condition (G_1) can not imply being numeroid, since a numeroid is a normaloid as pointed bout by Hildebrandt [5]. In the converse direction we shall show

Theorem 6. *There is a numeroid which does not satisfy the growth condition* (G_1).

Proof. Put

$$A=\frac{1}{2}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Then we have $\sigma(A)=\{1/2\}$ and $\|A\|\leqq 1$; hence D is a spectral set for A. Let B be the bilateral shift. Then by [4; Problem 68] $\sigma(B)$ is the unit circle C and $\bar{W}(B)=D$. Applying Theorem 3 for $T=A\oplus B$, we can conclude that T is a numeroid. Clearly, we have $\sigma(T)=C\cup\{1/2\}$. Furthermore, we have

$$A^{-1}=2\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}.$$

If we put

$$x=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix},$$

then we have

$$\|A^{-1}x\|=2\sqrt{2+1/2}>2.$$

If T satisfies the condition (G_1), then

$$2<\|A^{-1}\|\leqq\|T^{-1}\|\leqq\frac{1}{\operatorname{dist}(0,\sigma(T))}=2,$$

and this contradiction proves the theorem.

As a consequence of Theorem 6, we can prove the following theorem which is already established by Schreiber [9] using Toeplitz operators.

Theorem 7 (Schreiber). *There is a numeroid which is not a spectroid.*

Proof. By Theorem 6, there is an operator T which is a numeroid and does not satisfy the condition (G_1). On the other hand, every spectroid satisfies the condition (G_1). Hence T is not a spectroid.

Remark. In the proofs of Theorems 6 and 7, we used the bilateral shift in a contrast with Theorem 5. However, in a different point of view, we can give an another simple example: Let

$$A=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$B=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where the triangle with vertices λ_1, λ_2 and λ_3 contains the unit disc D. Since D is a spectral set for A and B is normal with $D\subset W(B)$, $T=A\oplus B$ is a numeroid by Theorem 3 and clearly non-normal. On the other hand, an operator satisfying the condition (G_1) is normal in a finite dimensional space. Hence T can not satisfy the condition (G_1).

This example shows also that there is a non-normal compact numeroid.

4. Appendix. Sz.-Nagy and Foiaş [10] introduced the following notion (without name):

Definition C. A point λ of a compact set S (in the plane) is called a *naked point* if there are λ_n and r_n such that

(i) $\{\mu ; |\mu-\lambda_n|<r_n\}\subset S^c$,

(ii) λ_n converges to λ as $n\to\infty$,

and

(iii) $\dfrac{|\lambda_n-\lambda|}{r_n}\to 1$ as $n\to\infty$,

where S^c is the complement of S.

Yoshino [11] introduced

Definition D. A point λ of S is *semi-bare* if there is a circle through λ such that no points of S lie inside the circle.

In [1], the following theorem is proved:

Theorem E (Berberian). *If T is an operator satisfying the condition* (G_1), *and if λ is a semi-bare point of $\sigma(T)$, then*

$$\ker(T-\lambda)=\ker(T^*-\lambda^*).$$

Since a semi-bare point is a naked point, the following theorem is an extension of Theorem E:

Theorem 8. *If T is an operator satisfying the condition* (G_1), *and if λ is a naked point of the spectrum $\sigma(T)$, then*

$$\ker(T-\lambda)=\ker(T^*-\lambda^*).$$

Proof. Translating if necessary, we can assume $\lambda=0$. By the hypothesis, there are λ_n and r_n such as

$$\{\mu\,;\,|\mu-\lambda_n|<r_n\}\subset\sigma(T)^c,\quad \lambda_n\to 0,\quad \frac{|\lambda_n|}{r_n}\to 1.$$

We can assume furthermore that $r_n=\operatorname{dist}(\lambda_n,\sigma(T))$. Put

$$\varepsilon_n=|\lambda_n|-r_n,$$

then we have

$$\frac{\varepsilon_n}{r_n}\to 0 \qquad (n\to\infty).$$

Define $a_n=e^{i\arg\lambda_n}$ and

$$W_n=a_n\lambda_n(T-\lambda_n)^{-1}.$$

Since T satisfies the condition (G_1), we have

$$\|W_n\|=|a_n|\,|\lambda_n|\,\|(T-\lambda_n)^{-1}\| \leqq |a_n|\,|\lambda_n|/|\lambda_n|=1.$$

In the below, we shall show the following three statements by which Theorem 7 follows:

(1) $W_nx\to x$ if and only if $W_n^*x\to x$,

(2) $Tx=0$ if and only if $W_nx\to x$,

and

(3) $T^*x=0$ if and only if $W_n^*x\to x$.

Suppose that $W_nx\to x$, then we have

$$\|W_n^*x-x\|=\|W_n^*x\|^2+\|x\|^2-2\operatorname{Re}(W_n^*x\,|\,x) \leqq 2[\|x\|^2-\operatorname{Re}(x\,|\,W_nx)]\to 0.$$

The converse implication can be proved similarly. Hence (1) is proved.

Suppose $Tx=0$. Then we have

$$\|W_nx-x\|\leqq\|x-W^{-1}x\| =\left\|x-\frac{1}{a_n\varepsilon_n-\lambda_n}(T-\lambda_n)x\right\| =\frac{\varepsilon_n}{r_n}\|x\|\to 0.$$

Conversely, suppose that $W_nx\to x$. Then we have

$$\|(T-a_ne_n)x\|=\|(T-\lambda_n)(1-W_n)x\| \leqq(\|T\|+\sup_n|\lambda_n|)\,\|W_nx-x\|\leftarrow 0.$$

Therefore $Tx=\lim_n a_n\varepsilon_nx=0$; hence (2) is proved.

Since we can prove (3) similarly, we have proved Theorem 8.

Remark. (1) We wish to point out that there exists a naked point which is not semi-bare. Let S be the territory being surrounded by

$$y=\pm 1 \qquad x=-1,$$

and the envelope of circles

$$\left(x-\frac{1}{t}\right)^2+y^2=\frac{1}{(t+1)^2} \qquad (t\geqq 1).$$

The origin is a naked point of S, it is not semi-bare.

(2) After the preparation of the present note, Mr. Nakamoto kindly informed us that Theorem 8 is also proved by T. Saito in his unpublished paper using a lemma of Sz.-Nagy and Foiaş.

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