101. *A Remark on Perturbation of m-accretive Operators in Banach Space*

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1. Introduction. Let X be a real Banach space with the norm denoted by $\|\cdot\|$. By definition a (possibly) multiple-valued operator A in X is *accretive* if for each $\lambda>0$ and $u, v \in D(A)$,

$$\|x-y\| \geq \|u-v\| \quad \text{whenever} \quad x \in (I+\lambda A)u, y \in (I+\lambda A)v.$$

An accretive operator A in X is said to be *m-accretive* if $R(I+A)=X$. For the notion of "multiple-valued" operator, we refer to T. Kato [6], § 2.

The purpose of the present paper is to give a criterion for the m-accretiveness of the sum of two m-accretive operators in X and then apply it to a certain nonlinear partial differential equation. Our result may be considered to constitute an analogue of the result of H. Brezis, M. G. Crandall and A. Pazy [1] for perturbation of maximal monotone sets.

2. A perturbation lemma. Let A and B be m-accretive operators in X. As usual we define the *Yosida approximation* $B_\varepsilon (\varepsilon>0)$ of B by

$$B_\varepsilon = \varepsilon^{-1}\{I-(I+\varepsilon B)^{-1}\},$$

which is a single-valued Lipschitz continuous operator defined on all of X. It is easily seen that $A+B_\varepsilon$ is again m-accretive and accordingly that for each $f \in X$ there exists a unique solution $u_\varepsilon \in D(A)$ of the equation

$$u_\varepsilon + y_\varepsilon + B_\varepsilon u_\varepsilon = f, \qquad y_\varepsilon \in A u_\varepsilon. \tag{2.1}$$

Lemma 1. *Assume that X is a real Banach space with the uniformly convex dual space X^* and that A and B are m-accretive operators in X such that $D(A) \cap D(B) \ni 0$. If for each fixed $f \in X$ $\|B_\varepsilon u_\varepsilon\|$ in* (2.1) *is bounded as ε tends to zero, then $A+B$ is m-accretive.*

We notice that if X^* is uniformly convex, the duality map F defined as

$$Fu = \{u^* \in X^* ; (u, u^*) = \|u\|^2 = \|u^*\|^2\}, \qquad u \in X,$$

is single-valued and is uniformly continuous on any bounded set (T. Kato [5]).

Proof of Lemma 1. The argument of the proof is standard (see Y. Kōmura [7] and T. Kato [5,6]). Since $D(A+B) \ni 0$, there is no loss of generality in assuming that $(A+B)0 \ni 0$ and we shall henceforth assume this. For each fixed $f \in X$, there exists a sequence $\{\varepsilon_n\}_{n\ge 1}$ such that $\varepsilon_n \downarrow 0$ and

(2.2) $$\text{w-}\lim_{n\to\infty} B_{\varepsilon_n} u_{\varepsilon_n} = z_0$$

exists in X. Since $\|u_{\varepsilon_n} - u_{\varepsilon_m}\| \le 2\|f\|$ and

$$\|(u_{\varepsilon_n} - u_{\varepsilon_m}) - \{(I+\varepsilon_n B)^{-1} u_{\varepsilon_n} - (I+\varepsilon_m B)^{-1} u_{\varepsilon_m}\}\| \le (\varepsilon_n + \varepsilon_m)\cdot \sup_{\nu\ge 1} \|B_{\varepsilon_\nu} u_{\varepsilon_\nu}\|$$

for $n, m = 1, 2, \cdots$, we have

$$\lim_{n,m\to\infty} \|F(u_{\varepsilon_n} - u_{\varepsilon_m}) - F((I+\varepsilon_n B)^{-1} u_{\varepsilon_n} - (I+\varepsilon_m B)^{-1} u_{\varepsilon_m})\| = 0.$$

On the other hand, by the accretiveness of A and B, we have

$$\begin{aligned}\|u_{\varepsilon_n} - u_{\varepsilon_m}\|^2 &= (u_{\varepsilon_n} - u_{\varepsilon_m}, F(u_{\varepsilon_n} - u_{\varepsilon_m})) \\ &\le -(B_{\varepsilon_n} u_{\varepsilon_n} - B_{\varepsilon_m} u_{\varepsilon_m}, F(u_{\varepsilon_n} - u_{\varepsilon_m})) \\ &\le -(B_{\varepsilon_n} u_{\varepsilon_n} - B_{\varepsilon_m} u_{\varepsilon_m}, F(u_{\varepsilon_n} - u_{\varepsilon_m}) \\ &\qquad - F((I+\varepsilon_n B)^{-1} u_{\varepsilon_n} - (I+\varepsilon_m B)^{-1} u_{\varepsilon_m}))\end{aligned}$$

for $n, m = 1, 2, \cdots$, which implies that $\lim_{n,m\to\infty} \|u_{\varepsilon_n} - u_{\varepsilon_m}\| = 0$. Hence we set

(2.3) $$\lim_{n\to\infty} u_{\varepsilon_n} = u_0.$$

In view of (2.1), (2.2) and (2.3), we obtain by Lemma 3.7 (a) and Lemma 4.5 in T. Kato [6] that $f \in R(I+A+B)$. Q.E.D.

3. Example. We denote by Ω a bounded domain in R^n with smooth boundary $\partial\Omega$, by $\mathcal{D}(\Omega)$ the Schwartz space and by $W_0^{k,p}(\Omega)$, $W^{k,p}(\Omega)$ the Sobolev spaces. Let β be an m-accretive operator in R such that $D(\beta) \ni 0$. We introduce the following m-accretive operator $\bar\beta$ in $L^p(\Omega)$:

$$D(\bar\beta) = \{u \in L^p(\Omega)\,;\ \text{for some } v \in L^p(\Omega),\ v(x) \in \beta(u(x)) \text{ a.e. in } \Omega\},$$
$$\bar\beta(u) = \{v \in L^p(\Omega)\,;\ v(x) \in \beta(u(x)) \text{ a.e. in } \Omega\} \text{ for } u \in D(\bar\beta).$$

Theorem 2. *We assume that* $1 < p < +\infty$ *and define an operator* Λ *in* $L^p(\Omega)$ *by*

$$D(\Lambda) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \cap D(\bar\beta),$$
$$\Lambda u = -\Delta u + \bar\beta(u) \text{ for } u \in (\Lambda).$$

Then Λ *is m-accretive.*

It is well known that $-\Delta$ with the domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is m-accretive in $L^p(\Omega)$.

Proof of Theorem 2. As in the proof of Lemma 1 we can assume that $\beta(0) \ni 0$. Let f be an arbitrary element of $L^p(\Omega)$ and let u_ε ($\varepsilon > 0$) be the unique solution of the equation: $u_\varepsilon - \Delta u_\varepsilon + \bar\beta_\varepsilon(u_\varepsilon) = f$. Then

(3.1) $(u_\varepsilon, F(\bar\beta_\varepsilon(u_\varepsilon))) + (-\Delta u_\varepsilon, F(\bar\beta_\varepsilon(u_\varepsilon))) + \|\bar\beta_\varepsilon(u_\varepsilon)\|^2 \le \|f\|\cdot\|\bar\beta_\varepsilon(u_\varepsilon)\|$.

Since $F(v) = v|v|^{p-2}/\|v\|^{p-2}$ for $v \in L^p(\Omega) \setminus \{0\}$, we have

(3.2) $$(u_\varepsilon, F(\bar\beta_\varepsilon(u_\varepsilon))) \ge 0.$$

Next we shall show that

(3.3) $$(-\Delta u_\varepsilon, F(\bar\beta_\varepsilon(u_\varepsilon))) \ge 0.$$

Let $\{u_\varepsilon^{(N)}\}_{N\geq 1}\subset\mathcal{D}(\Omega)$ be a sequence such that $\lim_{N\to\infty}\|u_\varepsilon^{(N)}-u_\varepsilon\|=0$ and define a sequence $\{\varphi_{(\nu)}\}_{\nu\geq 1}$ of monotone non-decreasing Lipschitz continuous functions on R by the following:

(i) If $2\leq p<+\infty$, $\quad \varphi_{(\nu)}(s)=s\,|s|^{p-2} \quad (s\in R)$.

(ii) If $1<p<2$, $\quad \varphi_{(\nu)}(s)=\begin{cases} s/|\nu|^{p-2} & (|s|\leq 1/\nu) \\ s\,|s|^{p-2} & (|s|>1/\nu). \end{cases}$

Then we can easily obtain that

$$(-\Delta(I-\lambda\Delta)^{-1}u_\varepsilon^{(N)}, \overline{\varphi}_{(\nu)}(\overline{\beta}_\varepsilon((I-\lambda\Delta)^{-1}u_\varepsilon^{(N)})))\geq 0 \quad \text{for each } \lambda>0.$$

Letting ν tend to infinity, we have by Lebesgue's theorem that

$$(-\Delta(I-\lambda\Delta)^{-1}u_\varepsilon^{(N)}, F(\overline{\beta}_\varepsilon((I-\lambda\Delta)^{-1}u_\varepsilon^{(N)})))\geq 0 \quad \text{for each } \lambda>0.$$

Letting N tend to infinity and then λ to zero, we obtain (3.3) since F is continuous. In view of (3.1), (3.2) and (3.3), we have $\|\overline{\beta}_\varepsilon(u_\varepsilon)\|\leq\|f\|$. Therefore by Lemma 1 A is m-accretive. Q.E.D.

Remark. (i) A is also "T-accretive" (B. Calvert [2]): $-A$ is "dispersive (s)" (Y. Konishi [8, 9]). (ii) Moreover $-A$ satisfies $D_3(e,0,+\infty)$ for any non-negative constant function e (K. Sato [10]): If we set $\varphi_e(f,g)=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(\|(f-e+\varepsilon g)^+\|-\|(f-e)^+\|)$ for $f,g\in L^p(\Omega)$, then for every $u,v\in D(A)$ with $(u-v-e)^+\neq 0$,

$$\varphi_e(u-v,x-y)\leq 0 \quad \text{whenever} \quad x\in -Au, y\in -Av.$$

(iii) If $\beta(0)\ni 0$, $-A|_{L^p(\Omega)^+}\in K_L(L^p(\Omega)^+)$(G. Da Prato [4]); where $L^p(\Omega)^+$ is the cone of all non-negative elements of $L^p(\Omega)$.

Applying Theorem 7.1 in T. Kato [6] and Theorem B in Y. Konishi [8] to the operator A in Theorem 2, we have

Corollary 3. *Assume that* $1<p<+\infty$ *and* $u_0\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\cap D(\overline{\beta})$. *Then the equation*

$$\begin{aligned} &0\in\partial u/\partial t-\Delta u+\overline{\beta}(u) && \text{in } \Omega\times(0,+\infty) \\ &u(x,t)=0 && \text{on } \partial\Omega\times(0,+\infty) \\ &u(x,0)=u_0(x) && \text{in } \Omega \end{aligned} \tag{3.4}$$

has a unique solution $u(x,t)\in C(0,+\infty;L^p(\Omega))$ *such that* $u(x,t)\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\cap D(\overline{\beta})$ *for every fixed* $t\geq 0$ *and* $\partial u/\partial t\in L^\infty(0,+\infty;L^p(\Omega))$. *Moreover if* u_1 *and* u_2 *are solutions of* (3.4) *such that* $u_1(x,0)\geq u_2(x,0)$ *a.e. in* Ω, *then* $u_1(x,t)\geq u_2(x,t)$ *a.e. in* Ω *for every fixed* $t\geq 0$.

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