100. On Power Semigroups

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(Comm. by Kenjiro SHODA, M. J. A., May 12, 1971)

1. If X is a semigroup, then the product of non-empty subsets of X can be defined in a natural way to produce a semigroup, which is called the power semigroup of X ([4]), and is denoted by $\mathfrak{T}(X)$. It is obvious that semigroups X and Y are isomorphic, then the power semigroups $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ are isomorphic. This note is devoted to the converse question: if $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ are isomorphic, must X and Y be isomorphic? We will answer this question for commutative semigroups whose ideals are all principal ideals. In the case for finite groups and chains, see [4].

2. By a partially ordered semigroup we mean a set X satisfying

(P1) X is a semigroup;

(P2) X is a partially ordered set under a relation $\leqq$;

(P3) $a \leqq b$ implies $ac \leqq bc$ and $ca \leqq cb$ for all $c \in X$ ([1], p. 153).

Let X and Y be partially ordered semigroups. By an o-isomorphism of X onto Y we mean a one-to-one mapping $\emptyset$ of X onto Y such that

(01) $\emptyset(ab) = \emptyset(a)\emptyset(b)$ for all $a, b \in X$;

(02) $a \leqq b$ in X if and only if $\emptyset(a) \leqq \emptyset(b)$ in Y.

3. Let X be a semigroup and $\mathfrak{T}(X)$ the set of all non-empty subsets of X. A binary operation is defined in $\mathfrak{T}(X)$ as follows: For A, $B \in \mathfrak{T}(X)$

$$AB = \{ab\,;\, a \in A, b \in B\}.$$

Then it is well-known and is easily seen that $\mathfrak{T}(X)$ is a semigroup. This semigroup $\mathfrak{T}(X)$ is called the power semigroup of X.

We define a relation $\leqq$ on $\mathfrak{T}(X)$ as follows; For $A, B \in \mathfrak{T}(X)$,

$$A \leqq B \text{ if and only if } A \subseteq B.$$

Then, as is well-known ([2], p. 132), $\mathfrak{T}(X)$ is a partially ordered set under this relation $\leqq$ satisfying the condition (P3), that is, $\mathfrak{T}(X)$ is a partially ordered semigroup.

Let $\mathfrak{J}(X)$ be the set of all ideals of a semigroup X and $\mathfrak{P}(X)$ the set of all principal ideals of X. Then clearly $\mathfrak{J}(X)$ is a subsemigroup of $\mathfrak{T}(X)$.

4. **Proposition 1.** *Let $\emptyset$ be an o-isomorphism of $\mathfrak{T}(X)$ onto $\mathfrak{T}(Y)$ and $\emptyset^*$ the restriction of $\emptyset$ on $\mathfrak{J}(X)$. Then $\emptyset^*$ maps $\mathfrak{J}(X)$ onto $\mathfrak{J}(Y)$.*

Therefore $\emptyset^*$ *is an isomorphism of* $\mathfrak{J}(X)$ *onto* $\mathfrak{J}(Y)$.

Proof. Since $\emptyset$ is an onto mapping, for $Y \in \mathfrak{T}(Y)$ there exists an element $B \in \mathfrak{T}(X)$ such that

$$\emptyset(B) = Y.$$

Let A be any element of $\mathfrak{T}(X)$. Then, since $\emptyset$ is an o-isomorphism of $\mathfrak{T}(X)$ onto $\mathfrak{T}(Y)$, we have

$$\emptyset(A)Y = \emptyset(A)\emptyset(B) = \emptyset(AB) \subseteq \emptyset(A)$$

and

$$Y\emptyset(A) = \emptyset(B)\emptyset(A) = \emptyset(BA) \subseteq \emptyset(A).$$

Thus we obtain that

$$\emptyset(\mathfrak{J}(X)) \subseteq \mathfrak{J}(Y).$$

Similarly, we have

$$\emptyset^{-1}(\mathfrak{J}(Y)) \subseteq \mathfrak{J}(X),$$

and so

$$\mathfrak{J}(Y) = \emptyset(\emptyset^{-1}(\mathfrak{J}(Y))) \subseteq \emptyset(\mathfrak{J}(X)).$$

Therefore we obtain that

$$\emptyset(\mathfrak{J}(X)) = \mathfrak{J}(Y),$$

which completes the proof of the proposition.

5. We denote by $[x]$ the principal ideal of a semigroup X generated by x of X. The following result is easily seen.

Proposition 2. *Let X be a commutative semigroup. Then*

$$[a][b] = [ab]$$

for every $a, b \in X$.

6. A semigroup X is called an IO-semigroup if $a \in [b]$ and $b \in [a]$ imply $a = b$. The definition and some properties concerning IO-semigroups are given by G. Szász [3].

Theorem 3. *Let X and Y be commutative IO-semigroups such that* $\mathfrak{P}(X) = \mathfrak{J}(X)$ *and* $\mathfrak{P}(Y) = \mathfrak{J}(Y)$. *If* $\mathfrak{T}(X)$ *and* $\mathfrak{T}(Y)$ *are o-isomorphic, then X and Y are isomorphic.*

Proof. Suppose that $\emptyset$ is an o-isomorphism of $\mathfrak{T}(X)$ onto $\mathfrak{T}(Y)$ and $\emptyset^*$ is the restriction of $\emptyset$ on $\mathfrak{J}(X)$. Consider the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{x^*} & \mathfrak{J}(X) & \subseteq & \mathfrak{T}(X) \\ {\scriptstyle h}\downarrow & & {\scriptstyle \emptyset^*}\downarrow & & \downarrow{\scriptstyle \emptyset} \\ Y & \xrightarrow[y^*]{} & \mathfrak{J}(Y) & \subseteq & \mathfrak{T}(Y) \end{array}$$

where

$$x^*(a) = [a] \quad \text{for every } a \in X,$$
$$y^*(b) = [b] \quad \text{for every } b \in Y.$$

Then it follows from Proposition 2 and the definition of the IO-semigroup that the mappings x^* and y^* are isomorphisms of X onto $\mathfrak{J}(X)$ and of Y onto $\mathfrak{J}(Y)$, respectively.

Moreover, since $\emptyset^*$ is an isomorphism of $\mathfrak{J}(X)$ onto $\mathfrak{J}(Y)$ by Proposition 1, we can prove that

$$h = y^{*-1} \circ \emptyset^* \circ x^*$$

gives an isomorphism of X onto Y. This completes the proof of the theorem.

The following corollary is the immediate consequence of Theorem 3.

Corollary 4. *Let X and Y be commutative IO-semigroups such that $\mathfrak{P}(X)=\mathfrak{I}(X)$ and $\mathfrak{P}(Y)=\mathfrak{I}(Y)$. If $\mathfrak{I}(X)$ and $\mathfrak{I}(Y)$ are isomorphic, then X and Y are isomorphic.*

References

[1] L. Fuchs: Partially Ordered Algebraic Systems. Pergamon Press (1963).

[2] E. C. Lyapin: Semigroups. Moscow (1960) (Russian).

[3] G. Szász: Über eine Quasiordnung in Halbgruppen. Publicationes Mathematicae, **13**, 43–46 (1966).

[4] T. Tamura and J. Shafer: Power semigroups. Mathematicae Japonicae, **12**, 25–32 (1967).