99. *Note on Simple Semigroups*

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1. By a *left* (*right*) *ideal* of a semigroup S we mean a non-empty subset X of S such that $SX\subseteq X$ $(XS\subseteq X)$. By a *two-sided ideal*, or simply *ideal*, we mean a subset of S which is both a left and a right ideal of S. A semigroup S is called *simple* if it contains no proper two-sided ideal. We denote by $[x]$ the principal ideal of S generated by x of S. A semigroup S is called *left* (*right*) *zero* if $xy=x$ $(xy=y)$ for every $x, y \in S$. Let $\mathfrak{T}(S)$ be the set of all non-empty subsets of a semigroup S. A binary operation is defined in $\mathfrak{T}(S)$ as follows: For X, $Y \in \mathfrak{T}(S)$,

$$XY=\{xy\,;\, x \in X, y \in Y\}.$$

Then it is easily seen that $\mathfrak{T}(S)$ is a semigroup.

Let $\mathfrak{J}(S)$ be the set of all ideals of a semigroup S and $\mathfrak{P}(S)$ the set of all principal ideals of S. It is clear that $\mathfrak{J}(S)$ is a subsemigroup of $\mathfrak{T}(S)$. The author proved in [2] that $\mathfrak{J}(S)$ is an idempotent semigroup if and only if $\mathfrak{P}(S)$ is an idempotent semigroup, and then both $\mathfrak{J}(S)$ and $\mathfrak{P}(S)$ are commutative. In this note we shall prove the following theorem:

Theorem 1. *Let S be a semigroup. Then S is a simple semigroup if and only if any one of the following conditions* (A)–(D) *holds*:

(A) $\mathfrak{J}(S)$ *is a left zero semigroup.*

(B) $\mathfrak{J}(S)$ *is a right zero semigroup.*

(C) $\mathfrak{P}(S)$ *is a left zero semigroup.*

(D) $\mathfrak{P}(S)$ *is a right zero semigroup.*

2. First we mention a result from our previous paper [2].

Lemma 2. *The following statements on a semigroup S are equivalent*:

(i) $X^2=X$ *for every* $X \in \mathfrak{J}(S)$.

(ii) $X\cap Y=XY$ *for every* $X, Y \in \mathfrak{J}(S)$.

(iii) $[x]^2=[x]$ *for every* $[x] \in \mathfrak{P}(S)$.

(iv) $[x]\cap[y]=[x][y]$ *for every* $[x], [y] \in \mathfrak{P}(S)$.

3. Proof of Theorem 1. Assume that S is simple, then it is clear that (A) holds. Conversely, if (A) holds, then, since $\mathfrak{J}(S)$ is an idempotent semigroup, it follows from (i), (ii) of Lemma 2 that

$$X=XY=X\cap Y$$

for every $X, Y \in \mathfrak{J}(S)$, and so

$$Y \subseteq X.$$

Thus we have

$$Y = X,$$

which implies that

$$X = S$$

for every $X \in \mathfrak{J}(S)$. This means that S is simple.

Similarly, we can prove that S is simple if and only if (B) holds.

Clearly (A) implies (C). Conversely, we assume that (C) holds. In order to prove that $X \subseteq XY$ for every $X, Y \in \mathfrak{J}(S)$, let $x \in X$ and $y \in Y$ be any elements of X and Y. Then, by the assumption (C) and by (iii), (iv) of Lemma 2, we have

$$[x] = [y].$$

Then it follows from (i), (ii) of Lemma 2 that

$$x \in [x] = [y] \subseteq Y,$$

and so

$$x \in X \cap Y = XY.$$

Hence we obtain that

$$X \subseteq XY$$

for every $X, Y \in \mathfrak{J}(S)$. Since $XY \subseteq X$, we have that (C) implies (A).

Similarly we can prove that (B) is equivalent to (D). This completes the proof of the theorem.

4. The following corollary can be easily seen.

Corollary 3. *Let S be a commutative semigroup. Then S is a group if and only if any one of the conditions* (A)–(D) *of Theorem* 1 *holds.*

5. Remark. For another characterisation of a simple semigroup by means of ideals, see [3].

References

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[2] N. Kuroki: On semigroups whose ideals are all globally idempotent. Proc. Japan Acad., **47**, 143–146 (1971).

[3] P. S. Venkatesan: On regular semigroups. Indian J. Math., **4**, 107–110 (1962).