98. On Distributive Sublattices of a Lattice*)

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In his note [1], B. Jónsson gave a necessary and sufficient condition that the sublattice generated by a subset H of a modular lattice should be distributive. This condition can be proved to be equivalent to the statement that $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$ for any a, b, c in H_1, where H_1 consists of all elements which can be written as a finite join or a finite meet of elements in H. The main purpose of this paper is to prove that in order for the sublattice generated by a subset H of a lattice to be distributive it is a necessary and sufficient condition that $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$ for any a, b, c in H_2, where H_2 consists of all elements which can be written as a finite join or a finite meet of elements in H_1.

Let $\langle H\rangle$ be a sublattice generated by a nonempty subset H of a lattice L. The finite join $\bigcup_{i=1}^m x_i$ of elements $x_1, x_2, \cdots, x_m$ in H is called a $\cup$-element in $\langle H\rangle$. The set of all $\cup$-elements in $\langle H\rangle$ is denoted by $H_\cup$ and dually the set of all $\cap$-elements in $\langle H\rangle$ by $H_\cap$. One of $\cup$- or $\cap$-elements in $\langle H\rangle$ is said to be a 1st-element in $\langle H\rangle$, and the set of all 1st-elements in $\langle H\rangle$ is denoted by H_1. The finite join $\bigcup_{i=1}^m x_i$ of $\cap$-elements $x_1, x_2, \cdots, x_m$ in $\langle H\rangle$ is called a $\cup\cap$-element in $\langle H\rangle$. The set of all $\cup\cap$-elements in $\langle H\rangle$ is denoted by $H_{\cup\cap}$ and dually the set of all $\cap\cup$-elements in $\langle H\rangle$ by $H_{\cap\cup}$. One of $\cup\cap$- or $\cap\cup$-elements in $\langle H\rangle$ is said to be a 2nd-element in $\langle H\rangle$, and the set of all 2nd-elements in $\langle H\rangle$ is denoted by H_2.

Two modular laws will be denoted by

μ: $(a\cap c)\cup(b\cap c)=((a\cap c)\cup b)\cap c$, and

μ^*: $(a\cup c)\cap(b\cup c)=((a\cup c)\cap b)\cup c$.

Four distributive laws will be denoted by

δ: $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$,

δ^*: $(a\cup c)\cap(b\cup c)=(a\cap b)\cup c$,

Δ: $\bigcup_{i=1}^m(x_i\cap y)=(\bigcup_{i=1}^m x_i)\cap y$, and

Δ^*: $\bigcap_{i=1}^m(x_i\cup y)=(\bigcap_{i=1}^m x_i)\cup y$.

Theorem 1. *Let $\langle H\rangle$ be the sublattice generated by a nonempty subset H of a lattice L. In order that $\langle H\rangle$ be distributive it is necessary and sufficient that Δ holds for any $x_i\in H$ $(i=1, 2, \cdots, m)$ and any $y\in H_\cap$ (or briefly Δ holds for H), and μ and μ^* hold for any $a, b, c\in H_2$*

*) Dedicated to Professor K. Asano on his sixtieth birthday.

(or briefly μ and μ^* hold for H_2).

Proof. This condition is obviously necessary. To prove that it is also sufficient, we first show that whenever X is any subset of L and μ and μ^* hold for X_2, Δ holds for X if and only if Δ^* holds for X. Δ^* clearly holds for $m=1$. Assuming that it holds for $m=k$, we consider the case in which $m=k+1$. Let $y=\bigcup_{j=1}^{m} y_j$, where $y_j \in H$. Then

$$\begin{aligned}
&\bigcap_{i=1}^{m} (x_i \cup y) \\
&\quad =(x_1 \cup y) \cap \bigcap_{i=2}^{m} (x_i \cup y) \\
&\quad =(x_1 \cup y) \cap \left(\left(\bigcap_{i=2}^{m} x_i\right) \cup y\right) \cdots \qquad \text{(by the hypothesis)} \\
&\quad =\left(\left(x_1 \cup \bigcup_{j=1}^{n} y_j\right) \cap \bigcap_{i=1}^{m} x_i\right) \cup y \cdots \qquad (\text{by } \mu^* \text{ for } X_2) \\
&\quad =\left(x_1 \cap \bigcap_{i=2}^{m} x_i\right) \cup \bigcup_{j=1}^{n} \left(y_j \cap \bigcap_{i=2}^{m} x_i\right) \cup y \cdots \qquad (\text{by } \Delta \text{ for } X) \\
&\quad =\left(\bigcap_{i=1}^{m} x_i\right) \cup \left(y \cap \bigcap_{i=2}^{m} x_i\right) \cup y \cdots \qquad (\text{by } \Delta^* \text{ for } X) \\
&\quad =\left(\bigcap_{i=1}^{m} x_i\right) \cup y.
\end{aligned}$$

Thus Δ^* is implied by Δ. By dualizing the proof we obtain that Δ is implied by Δ^*.

Now suppose H satisfies the condition of the theorem. Let S be the family of all subsets X of L satisfying the following three conditions:

(a) $X \supseteq H$,

(b) Δ and Δ^* hold for X, and

(c) μ and μ^* hold for X_2.

$\boldsymbol{S}$ is, obviously, partly ordered by the set inclusion. Then it is easily verified that the set union of any chain of sets belonging to $\boldsymbol{S}$ also has the properties (a), (b) and (c). From this, by applying the Zorn's lemma, $\boldsymbol{S}$ has at least one maximal element Z. Thus $Z \supseteq H$, Δ and Δ^* hold for Z and μ and μ^* hold for Z_2.

Suppose $u, v \in Z$ and let $Y = Z \vee \{u \cap v\}$. ($\vee$ is the set union.) Clearly, $Y \supseteq H$. In order to show that μ and μ^* hold for any $a, b, c \in Y_2$, we shall prove that $a \in Y_2$ implies $a \in Z_2$.

(1) The case in which $a \in Y_{\cup\cap}$:

$Y_{\cap} = Z_{\cap}$, so $a \in Y_{\cup\cap} = Z_{\cup\cap} \subseteq Z_2$.

(2) The case in which $a \in Y_{\cap\cup}$:

In this case a is represented as $a=\bigcap_{i=1}^{m} x_i$, where $x_i=\bigcup_{j(i)=1}^{n(i)} y_{j(i)} \in Y_{\cup}$ and $y_{j(i)} \in Y$. If none of the elements $y_{j(i)}$ equal $u \cap v$, then $a \in Z_2$. If some $y_{j(i)}$ equals $u \cap v$, then there exists x_i' such that $x_i = (u \cap v) \cup x_i'$, $x_i' = \bigcup_{j(i)=1}^{n'(i)} y'_{j(i)} \in Y_{\cup}$ and none of $y'_{j(i)}$ equal $u \cap v$. Thus $x_i' \in Z_{\cup}$.

$$x_i=(u\cap v)\cup x_i'$$
$$=(u\cup x_i')\cap(v\cup x_i')\cdots \qquad \text{(by } \Delta^* \text{ for } Z)$$

Since $u\cup x_i', v\cup x_i'\in Z_\cup$, each $x_i\in Z_{\cap\cup}$. Therefore $a=\bigcap_{i=1}^m x_i\in Z_{\cap\cup}\subseteq Z_2$.

Now we shall prove that μ and μ^* hold whenever $a, b, c\in Y_2$. If $a, b, c\in Y_2$ then $a, b, c\in Z_2$. Hence μ and μ^* hold for $a, b, c\in Z_2$, so also for $a, b, c\in Y_2$.

Next, in order to show that Δ holds whenever $x_i\in Y$ and $y\in Y_\cap$, we need only consider the essential case in which $x_2, \cdots, x_m\in Z$, $x_1=u\cap v$ and $y\in Z_\cap$, since $Z_\cap=Y_\cap$. Let $y=\bigcap_{j=1}^n y_j$, where $y_j\in Z$. Then we have

$$\bigcup_{i=1}^m (x_i\cap y)$$
$$=(x_1\cap y)\cup\bigcup_{i=2}^m (x_i\cap y)$$
$$=(x_1\cap y)\cup\left(\left(\bigcup_{i=2}^m x_i\right)\cap y\right)\cdots \qquad \text{(by } \Delta \text{ for } Z)$$
$$=\left(\left(u\cap v\cap\bigcap_{j=1}^n y_j\right)\cup\bigcup_{i=2}^m x_i\right)\cap y\cdots \qquad \text{(by } \mu \text{ for } Z_2)$$
$$=\left(u\cup\bigcup_{i=2}^m x_i\right)\cap\left(v\cup\bigcup_{i=2}^m x_i\right)\cap\bigcap_{j=1}^n\left(y_j\cup\bigcup_{i=2}^m x_i\right)\cap y\cdots \quad \text{(by } \Delta^* \text{ for } Z)$$
$$=\left((u\cap v)\cup\bigcup_{i=2}^m x_i\right)\cap\left(y\cup\bigcup_{i=2}^m x_i\right)\cap y\cdots \qquad \text{(by } \Delta^* \text{ for } Z)$$
$$=\left(\bigcup_{i=1}^m x_i\right)\cap y.$$

Thus Δ is true for Y. By our preliminary remark, so is Δ^*.

Therefore $Y\in \boldsymbol{S}$. On the other hand, since $Z\subseteq Y$ and Z is a maximal element in $\boldsymbol{S}$, $Z=Y$. Thus $u\cap v\in Z$ for any element $u, v\in Z$. Similarly $u\cup v\in Z$ for any element $u, v\in Z$, so that Z is a sublattice of L. Furthermore Z is distributive, because

$$(x_1\cap y)\cup(x_2\cap y)$$
$$=(x_1\cup x_2)\cap y\cdots \qquad \text{(by } \Delta \text{ for } Z)$$

$\langle H\rangle$ is therefore a sublattice of the distributive lattice Z, so that $\langle H\rangle$ is distributive.

Theorem 2. *Let $\langle H\rangle$ be the sublattice generated by a nonempty subset H of a lattice L. The following five statements are equivalent*:

(1) *$\langle H\rangle$ is distributive.*

(2) *δ holds for any $a, b, c\in H_2$.*

(3) *Δ holds for any $x_i\in H_\cap$ $(i=1, 2, \cdots, m)$ and any $y\in H_2$.*

(4) *δ and δ^* hold for any $a, b, c\in H_2$.*

(5) *Δ holds for any $x_i\in H$ $(i=1, 2, \cdots, m)$ and any $y\in H_\cap$, and μ and μ^* hold for any $a, b, c\in H_2$.*

Proof. (1)$\Rightarrow$(2): This implication is evident.

(2)$\Rightarrow$(3): We use induction on the number m. For any $x_i\in H_\cap$ and any $y\in H_2$,

$$\begin{aligned}
&\left(\bigcup_{i=1}^{m} x_i\right) \cap y \\
&\quad = \left(\bigcup_{i=1}^{m-1} x_i \cup x_m\right) \cap y \\
&\quad = \left(\left(\bigcup_{i=1}^{m-1} x_i\right) \cap y\right) \cup (x_m \cap y) \cdots \qquad \text{(by } \delta \text{ for } H_2\text{)} \\
&\quad = \bigcup_{i=1}^{m-1} (x_i \cap y) \cup (x_m \cap y) \cdots \qquad \text{(by the hypothesis)} \\
&\quad = \bigcup_{i=1}^{m} (x_i \cap y).
\end{aligned}$$

(3)⇒(4): By applying (3) twice, we prove that for any $x_i, y_j \in H$ ($i=1, 2, \cdots, m$: $j=1, 2, \cdots, n$),

$$\begin{aligned}
&\left(\bigcup_{i=1}^{m} x_i\right) \cap \left(\bigcup_{j=1}^{n} y_j\right) \\
&\quad = \bigcup_{i=1}^{m} \left(x_i \cap \left(\bigcup_{j=1}^{n} y_j\right)\right) \cdots \qquad \text{(by } \Delta \text{ in the condition (3))} \\
&\quad = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (x_i \cap y_j) \cdots \qquad \text{(by } \Delta \text{ in the condition (3)).}
\end{aligned}$$

Hence, by induction on number r,

$$\bigcap_{i=1}^{r} \left(\bigcup_{j(i)=1}^{n(i)} x_{j(i)}\right) = \bigcup_{j(1)=1}^{n(1)} \cdots \bigcup_{j(r)=1}^{n(r)} \left(\bigcap_{i=1}^{r} x_{ij(i)}\right)$$

for any $x_{ij(i)} \in H$ ($i=1, 2, \cdots, r$: $j(i)=1, 2, \cdots, n(i)$). Thus we have the following lemma.

Lemma. (3) *implies* $H_2 = H_{\cup\cap}$.

Now we shall show that δ holds for any $a, b, c \in H_2$. By Lemma, there exists x_i ($i=1, 2, \cdots, m+n$) in $H_\cap$ such that $a = \bigcup_{i=1}^{m} x_i$ and $b = \bigcup_{i=m+1}^{m+n} x_i$.

$$\begin{aligned}
&(a \cup b) \cap c \\
&\quad = \left(\bigcup_{i=1}^{m+n} x_i\right) \cap c \\
&\quad = \bigcup_{i=1}^{m+n} (x_i \cap c) \cdots \qquad \text{(by } \Delta \text{ in the condition (3))} \\
&\quad = \bigcup_{i=1}^{m} (x_i \cap c) \cup \bigcup_{i=m+1}^{m+n} (x_i \cap c) \\
&\quad = (a \cap c) \cup (b \cap c) \cdots \qquad \text{(by } \Delta \text{ in the condition (3))}
\end{aligned}$$

Next, we shall show that δ^* holds for any $a, b, c \in H_2$. By Lemma, there exists x_i ($i=1, 2, \cdots, m+n$) in $H_\cap$ such that $b = \bigcup_{i=1}^{m} x_i$ and $c = \bigcup_{i=m+1}^{m+n} x_i$. Thus $d = b \cup c = \bigcup_{i=1}^{m+n} x_i \in H_2$, and

$$\begin{aligned}
&(a \cup c) \cap (b \cup c) \\
&\quad = (a \cup c) \cap d \\
&\quad = (a \cap d) \cup (c \cap d) \cdots \qquad \text{(by } \delta \text{ for } H_2\text{)} \\
&\quad = (a \cap (b \cup c)) \cup (c \cap (b \cup c)) \\
&\quad = (a \cap b) \cup (a \cap c) \cup c \cdots \qquad \text{(by } \delta \text{ for } H_2\text{)} \\
&\quad = (a \cap b) \cup c.
\end{aligned}$$

(4)$\Rightarrow$(5): (4) implies (2) and (2) implies (3). (3) implies Δ for any $x_i \in H$ $(i=1, 2, \cdots, m)$ and any $y \in H_\cap$. Next, we shall show that μ holds for any $a, b, c \in H_2$.

$$((a \cup c) \cap b) \cup c$$
$$=(a \cap b) \cup (c \cap b) \cup c \cdots \qquad \text{(by } \delta \text{ for } H_2)$$
$$=(a \cap b) \cup c$$
$$=(a \cup c) \cap (b \cup c) \cdots \qquad \text{(by } \delta^* \text{ for } H_2).$$

In a similar way we can prove that μ^* holds for any $a, b, c \in H_2$.
(5)$\Rightarrow$(1): This implication is proved in Theorem 1.

Theorem 3. *Let $\langle H \rangle$ be the sublattice generated by a nonempty subset H of a modular lattice M. The following three statements are equivalent.*

(1) *$\langle H \rangle$ is distributive.*

(2) *δ holds for any $a, b, c \in H_1$.*

(3) *Δ holds for any $x_i \in H$ $(i=1, 2, \cdots, m)$ and any $y \in H_\cap$.*

Proof. The implication (1)$\Rightarrow$(2) is evident.

The implication (2)$\Rightarrow$(3) can be proved similarly to the proof (2) $\Rightarrow$(3) in Theorem 2.

The implication (3)$\Rightarrow$(1) is the original form of Jónsson's theorem in [1].

References

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[2] G. Birkhoff: Lattice Theory. Amer. Math. Soc. Colloquium Publication, **25**. New York.

[3] G. Szasz: Introduction to Lattice Theory. Academic Press, New York and London.