97. On the Relation between the Positive Definite Quadratic Forms with the Same Representation Numbers

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1. In this note we investigate the relation between the positive definite integral quadratic forms with the same representation numbers.

2. A positive definite $n\times n$ matrix $A=(a_{ij})$ is called even positive if all a_{ij} are integers and all a_{ii} are even integers; then we put $\vartheta(\tau, A) = \sum_{\xi\in Z^n} e^{\pi i A[\xi]\tau}$.

For an even positive $2k\times 2k$ matrix A we define the level of A by the smallest natural number N such that NA^{-1} is also even positive; then N divides $\det A$ and $\det A$ divides N^{2k}.

An even positive ternary matrix $\begin{pmatrix} 2a & g & f \\ g & 2b & e \\ f & e & 2c \end{pmatrix}$, which is denoted by $[a, b, c, e, f, g]$ for brevity, is called reduced in the sense of Seeber and Eisenstein if the following conditions are satisfied:

1) e, f, g are all positive or all non-positive.

2) $a\leq b\leq c, a+b+e+f+g\geq 0$.

3) $|f|\leq a, |g|\leq a, |e|\leq b$.

4) If $a=b, |e|\leq|f|$; if $b=c, |f|\leq|g|$; if $a+b+e+f+g=0$, $2a+2f+g\leq 0$.

5) For $e, f, g\leq 0$: if $a=-g, f=0$; if $a=-f, g=0$; if $b=-e, g=0$.

6) For $e, f, g>0$: if $a=g, f\leq 2e$; if $a=f, g\leq 2e$; if $b=e, g\leq 2f$.

We say that two matrices A, B are equivalent if $A={}^tTBT$ holds for some integral matrix T with determinant ± 1.

3. Theorem 1. *Assume that $\vartheta(\tau, A)=\vartheta(\tau, B)$ holds for two even positive matrices A, B. Then the following assertions* i), ii), iii) *and* iv) *are true.*

i) *There exists a matrix T with rational numbers as entries such that ${}^tTAT=B$ holds.*

ii) *In case that A is $2k\times 2k$ matrix, A and B belong to the same genus if the level N of A is* odd *or* $N\equiv 2 \bmod 4$.

iii) *In case that A is $(2k+1)\times(2k+1)$ matrix, A and B belong to the same genus if* $\det A=2^t r$ *holds, where $t\leq 4$ and r is odd.*

iv) *If A is $n\times n$ matrix with $n\leq 4$, A and B always belong to the same genus.*

It is likely that for two even positive matrices A, B, A and B always belong to the same genus if only $\vartheta(\tau, A)=\vartheta(\tau, B)$ holds, and that i) is also true for (real) positive matrices A, B.

As an application of the method of the proof of Theorem 1 we obtain

Theorem 2. *Assume that A and B are even positive $n \times n$ matrices with $\det A=\det B$. Then A and B belong to the same genus if the level of A is equal to the level of B and its value is 1 or a prime integer in case of even n, and $\det A=2p$ in case of odd n, where p is 1 or an odd prime integer.*

Theorem 3. *For two even positive ternary matrices A, B, A and B are equivalent if $\vartheta(\tau, A)=\vartheta(\tau, B)$ and at least one of the following conditions hold.*

i) *$\vartheta(\tau, A)$ has the Fourier expansion $1+a_1 e^{2\pi i\tau}+\cdots$ with $a_1 \neq 0$.*

ii) *A is a diagonal matrix. (B is not necessarily diagonal.)*

iii) *$A=[a, b, c, e, f, g]$ is a reduced matrix in the sense of Seeber and Eisenstein with $a+b<c$, $b \geq 4a$ and $|e| \leq b/2$.*

iv) *$A=[a, b, c, e, f, g]$ is a reduced matrix in the sense of Seeber and Eisenstein and $B=[a', b', c', e', f', g']$ is also reduced in the sense of Seeber and Eisenstein with $a'+b'<c'$, $b' \geq 2a'$ and either $e, e'>0$ or $e, e' \leq 0$.*

There is an example of even positive matrices A, B with 16 variables which are not equivalent, although $\vartheta(\tau, A)=\vartheta(\tau, B)$ holds (Witt). But it is always true in the binary case that $\vartheta(\tau, A)=\vartheta(\tau, B)$ implies the equivalence of A and B. This seems to be true also in the ternary case (even the quaternary real positive case).

The proof of Theorems 1, 2 are based on the theorem of Minkowski (p.p. 136–142 of [1]) and the following formula

$$\lim_{\tau \to i\infty} (c\tau+d)^{-(n/2)} \vartheta\left(\frac{a\tau+b}{c\tau+d}, A\right) = e^{-(\pi i/4)} c^{-(n/2)} \sqrt{\det A}^{-1} \sum_{\xi \bmod c} e^{\pi i (a/c) A[\xi]}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \boldsymbol{Z})$ with $c>0$.

In the proof of Theorem 1, it is shown that a genus for an even positive $n \times n$ matrix ($n \leq 4$) is completely determined by the determinant and Gaussian sums $\sum_{\xi \bmod c} e^{\pi i (a/c) A[\xi]}$ (a and $c (\neq 0)$ run over all integers). But this is not true for $n \geq 5$. For example

$$2\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 28 \end{pmatrix} \quad \text{and} \quad 2\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix}$$

have the same determinant and Gaussian sums, but A and B do not belong to the same genus.

Remark. The m-th coefficient in the Fourier expansion (with respect to $e^{2\pi i\tau}$) of $\vartheta(\tau, A)$ is the number of the vectors in the lattice $\boldsymbol{Z}^n$ in $\boldsymbol{R}^n$ which have the distance m from the origin with respect to the metric $(x, y) = (1/2)^t xAy$.

Detailed proof will appear elsewhere.

Reference

[1] H. Minkowski: Grundlagen für eine Theorie der quadratischen Formen mit ganzzahligen Koeffizienten, Gesammelte Abhandlungen 1. Leipzig, 3–144 (1911).