123. On the Existence of Solutions for System of Linear Partial Differential Equations with Constant Coefficients

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This paper is on the extension of a theorem by J. F. Treves (Lectures on linear partial differential equations with constant coefficients) for single linear partial differential equation to the case of system, which owes a great deal to the suggestions of Prof. Mitio Nagumo.

Let $\mathfrak{A}$ be a non-commutative algebra with unit over the complex numbers C, and let $[A,B]=AB-BA$ for all $A,B\in\mathfrak{A}$. Let $A_1, \cdots, A_n, B_1, \cdots, B_n$ be $2n$ elements of the algebra $\mathfrak{A}$, satisfying the following commutation relations:

(1) $[A_j, A_k]=[B_j, B_k]=0$ for $1\leq j, k\leq n$. $[A_j, B_k]=0$ for $j\neq k$.

(2) $[A_j, B_j]=I$ (unit element of $\mathfrak{A}$) for $1\leq j\leq n$.

Let $P(X)=P(X_1, \cdots, X_n)$ be a polynomial with complex coefficients, and p be a multi-index $(p_1, \cdots, p_n)$ of n integers ≥ 0, and let

$$P^{(p)}(X)=\left(\frac{\partial}{\partial X_1}\right)^{p_1}\cdots\left(\frac{\partial}{\partial X_n}\right)^{p_n}P(X_1, \cdots, X_n).$$

Lemma 1 (by lecture note of Treves). *Let $P(X), Q(X)$ be the polynomials in n letters with complex coefficients, then*

$$Q(B)P(A)=\sum_p \frac{(-1)^{|p|}}{p!}P^{(p)}(A)Q^{(p)}(B),$$

where $A=(A_1, \cdots, A_n), B=(B_1, \cdots, B_n)$ satisfying the above commutation relations (1), (2), *and* $|p|=p_1+\cdots+p_n, p!=p_1!\cdots p_n!$.

Lemma 2. *Let $\boldsymbol{P}(X), \boldsymbol{Q}(X)$ be arbitrary square matrix of (m,m)-type such that its elements are polynomials in n letters with complex coefficients, then*

$${}^t(\boldsymbol{Q}(B)\boldsymbol{P}(A))=\sum_p \frac{(-1)^{|p|}}{p!}{}^t\boldsymbol{P}^{(p)}(A)\,{}^t\boldsymbol{Q}^{(p)}(B).$$

Proof. This lemma follows immediately by substituting the equality in Lemma 1.

Now, assume that $\mathfrak{A}$ is an algebra of linear mappings $\mathcal{D}\to\mathcal{D}$, where $\mathcal{D}$ is the linear space of infinitely differentiable complex valued functions on $\boldsymbol{R}^n$ with compact support. Let $\mathcal{L}_2=L_2\times\cdots\times L_2$, and inner product of $\mathcal{L}_2$ is defined by $(\boldsymbol{f},\boldsymbol{g})_{\mathcal{L}_2}=\sum_{i=1}^m (f_i, g_i)_{L_2}$ for $\boldsymbol{f}=(f_1, \cdots, f_m)$,

$\boldsymbol{g}=(g_1, \cdots, g_m)$. Let $A^*=(A_1^*, \cdots, A_n^*), B=-A^*$, where A_j^* is the adjoint of A_j for the inner product of L_2 on $\mathcal{D}$. Further, assume that the commutation relations (1), (2) hold.

Put

$$\boldsymbol{Q}(X)={}^t\bar{\boldsymbol{P}}(-X)=\left(\bar{P}_{ij}(-X)\begin{matrix} i\rightarrow 1, \cdots, m \\ i\downarrow 1, \cdots, m\end{matrix}\right),$$

where $\bar{P}$ is the polynomial with replaced coefficients in P by its complex conjugate. Since $\boldsymbol{P}(A)^*={}^t\bar{\boldsymbol{P}}(A^*), \boldsymbol{Q}(B)=\boldsymbol{P}(A)^*$. On the other hand, since $\boldsymbol{Q}^{(p)}(X)=(-1)^{|p|}\,{}^t\bar{\boldsymbol{P}}^{(p)}(-X), \boldsymbol{Q}^{(p)}(B)=(-1)^{|p|}\boldsymbol{P}^{(p)}(A)^*$.
Hence, by Lemma 2,

$${}^t(\boldsymbol{P}(A)^*\boldsymbol{P}(A))=\sum_p \frac{1}{p\,!}\,{}^t\boldsymbol{P}^{(p)}(A)\,{}^t\boldsymbol{P}^{(p)}(A)^*.$$

Since

$$({}^t(\boldsymbol{P}(A)^*\boldsymbol{P}(A))\varphi, \varphi)_{\mathcal{L}_2}=\sum_p \frac{1}{p\,!}({}^t\boldsymbol{P}^{(p)}(A)\,{}^t\boldsymbol{P}^{(p)}(A)^*\varphi, \varphi)_{\mathcal{L}_2}$$

for every $\varphi\in(\mathcal{D})^m=\mathcal{D}\times\cdots\times\mathcal{D}$,

$$(\bar{\boldsymbol{P}}(A^*)\varphi, \bar{\boldsymbol{P}}(A^*)\varphi)_{\mathcal{L}_2}=\sum_p \frac{1}{p\,!}({}^t\boldsymbol{P}^{(p)}(A)^*\varphi, {}^t\boldsymbol{P}^{(p)}(A)^*\varphi)_{\mathcal{L}_2}.$$

Replacing A^* by A in the above equality, and since $\boldsymbol{P}$ is an arbitrary square matrix, replacing $\bar{\boldsymbol{P}}$ by $\boldsymbol{P}$,

$$\begin{aligned}(\boldsymbol{P}(A)\varphi, \boldsymbol{P}(A)\varphi)_{\mathcal{L}_2}&=\sum_p \frac{1}{p\,!}({}^t\bar{\boldsymbol{P}}^{(p)}(A^*)^*\varphi, {}^t\bar{\boldsymbol{P}}^{(p)}(A^*)^*\varphi)_{\mathcal{L}_2}\\ &=\sum_p \frac{1}{p\,!}(\boldsymbol{P}^{(p)}(A)\varphi, \boldsymbol{P}^{(p)}(A)\varphi)_{\mathcal{L}_2},\end{aligned}$$

namely,

$$\|\boldsymbol{P}(A)\varphi\|^2_{\mathcal{L}_2}=\sum_p \frac{1}{p\,!}\|\boldsymbol{P}^{(p)}(A)\varphi\|^2_{\mathcal{L}_2}$$

for every ${}^\forall\varphi\in(\mathcal{D})^m$.
Therefore the following lemma holds immediately from the above equality.

Lemma 3. *For every $\varphi\in(\mathcal{D})^m$, there exists a constant C such that $\|\boldsymbol{P}^{(p)}(A)\varphi\|^2_{\mathcal{L}_2}\leqq C\|\boldsymbol{P}(A)\varphi\|^2_{\mathcal{L}_2}$.*

Let $R^n\ni X=(X_1, \cdots, X_n)$, $D_j=\dfrac{\partial}{\partial X_j}(1\leqq j\leqq n)$ and let $t_1, \cdots, t_n$ be real numbers, all different from zero. Put $E(t,x)=\exp\dfrac{1}{2}(t_1^2x_1^2+\cdots+t_n^2x_n^2)$. Let $m\times m$ matrix $\boldsymbol{E}(t,x)=\begin{pmatrix}E(t,x) & & \\ & \ddots & \\ & & E(t,x)\end{pmatrix}$, then, for an arbitrary square matrix $\boldsymbol{P}(X)$ such that its elements are polynomials, the following theorem holds from Lemma 3.

Theorem 1. *There exists a constant C such that*

$$\|\boldsymbol{E}(t,x)\boldsymbol{P}^{(p)}(D)\varphi\|^2_{\mathcal{L}_2}\leqq C'\|\boldsymbol{E}(t,x)\boldsymbol{P}(D)\varphi\|^2_{\mathcal{L}_2}$$

for every $\varphi \in (\mathcal{D})^m$.

Proof. Let $A_j = \frac{1}{\sqrt{2}}(t_j^{-1}D_j - t_jX_j)$, then the adjoint of A_j is $A_j^* = -\frac{1}{\sqrt{2}}(t_j^{-1}D_j + t_jX_j)$. Put $B_j = -A_j^*$. The commutation relations (1), (2) are satisfied for $A = (A_1, \cdots, A_n), B = (B_1, \cdots, B_n)$. Let $\boldsymbol{P}_t(X) = \boldsymbol{P}(\sqrt{2}\,t_1X_1, \cdots, \sqrt{2}\,t_nX_n)$, then $\boldsymbol{P}_t^{(p)}(X) = \sqrt{2}^{\,|p|}t^p\boldsymbol{P}^{(p)}(\sqrt{2}\,t_1X_1, \cdots, \sqrt{2}\,t_nX_n)$. By substituting A_j for X_j, $\boldsymbol{P}_t^{(p)}(A) = \sqrt{2}^{\,|p|}t^p\boldsymbol{P}^{(p)}(D - t^2X)$, where $D - t^2X = (D_1 - t_1^2X_1, \cdots, D_n - t_n^2X_n)$. By applying Lemma 3 to this square matrix $\boldsymbol{P}_t(A)$,

$$2^{|p|}t^{2p}\|\boldsymbol{P}^{(p)}(D - t^2X)\varphi\|^2_{\mathcal{L}_2} \leqq C\|\boldsymbol{P}(D - t^2X)\varphi\|^2_{\mathcal{L}_2} \tag{1}$$

On the other hand, for every $\varphi \in (\mathcal{D})^m$,

$$\boldsymbol{P}(D - t^2X)\boldsymbol{E}(t, X)\varphi(X) = \left(\sum_{j=1}^m P_{ij}(D - t^2X)E(t, X)\varphi_j(X) \qquad i \downarrow 1, \cdots, m\right)$$

$$= \left(\sum_{j=1}^m E(t, X)P_{ij}(D)\varphi_j(X) \qquad i \downarrow 1, \cdots, m\right) = \boldsymbol{E}(t, X)\boldsymbol{P}(D)\varphi(X).$$

Hence, by applying above estimate to $\boldsymbol{E}(t, X)\varphi(X)$,

$$2^{|p|}t^{2p}\,\|\boldsymbol{E}(t, X)\boldsymbol{P}^{(p)}(D)\varphi\|^2_{\mathcal{L}_2} \leqq C\,\|\boldsymbol{E}(t, X)\boldsymbol{P}(D)\varphi\|^2_{\mathcal{L}_2}$$

for every $\varphi \in (\mathcal{D})^m$. (q.e.d.)

This estimate is essential in the next theorem.

Let $\mathcal{H}_+(t)$ be the linear space of m-tuples measurable functions $\boldsymbol{f}(X)$ on R^n such that $\boldsymbol{E}(t, X)\boldsymbol{f}(X) \in \mathcal{L}_2$, which is provided with the inner product, and the norm,

$$(\boldsymbol{f}, \boldsymbol{g})_{+,t} = \sum_{i=1}^m \int E^2(t, x)f_i(x)\overline{g_i(x)}dX, \qquad \|\boldsymbol{f}\|_{+,t} = \sqrt{(\boldsymbol{f}, \boldsymbol{f})_{+,t}}.$$

Then $\mathcal{H}_{+t}(t)$ is a Hilbert space. Similarly, the Hilbert space $\mathcal{H}_-(t)$ is defined by the linear space of m-tuples measurable functions $\boldsymbol{f}(X)$ on R^n such that $\boldsymbol{E}^{-1}(t, X)\boldsymbol{f}(X) \in \mathcal{L}_2$, with the inner product

$$(\boldsymbol{f}, \boldsymbol{g})_{-,t} = \sum_{i=1}^m \int E^{-2}(t, x)f_i(x)\overline{g_i(x)}dX.$$

The bilinear form $\langle \boldsymbol{f}, \boldsymbol{g}\rangle$ on $\mathcal{H}_+(t) \times \mathcal{H}_-(t)$ is defined by the following: $\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \sum_{i=1}^m \int f_i(x)g_i(x)dX$.

Then, $\mathcal{H}_-(t)$ is the dual space of $\mathcal{H}_+(t)$.

Estimate in Theorem 1 can be given by $\mathcal{H}_+(t)$-norm

$$\|\boldsymbol{P}^{(p)}(D)\varphi\|^2_{+,t} \leqq C\,\|\boldsymbol{P}(D)\varphi\|^2_{+,t} \qquad \text{for every } \varphi \in (\mathcal{D})^m.$$

Henceforth, let $\boldsymbol{P}(X)$ be the square matrix

$$\begin{pmatrix} P_{ij} & \begin{matrix} i \downarrow 1, \cdots, m \\ j \rightarrow 1, \cdots, m \end{matrix} \end{pmatrix}$$

satisfying the following conditions:

"*There exists a multi-index r such that* $\boldsymbol{P}^{(r)}(X) = \boldsymbol{C} =$ *constant matrix and* $\boldsymbol{C}$ *has an inverse matrix* $\boldsymbol{C}^{-1}$."

Let this condition be named "condition (C)".

Hence, for this multi-index r.

$$\|\varphi\|^2_{+,t} \leqq C\,\|P(D)\varphi\|^2_{+,t} \qquad \text{for every } \varphi \in (\mathcal{D})^m. \tag{3}$$

$P(D)(\mathcal{D})^m = \{P(D)\varphi\,;\, \varphi \in (\mathcal{D})^m\}$ is a linear subspace of $\mathcal{H}_+(t)$. As $P(D)(\mathcal{D})^m$ is provided with the $\mathcal{H}_+(t)$-norm, linear mapping $P(D)\varphi \to \varphi$ is continuous from $P(D)(\mathcal{D})^m$ into $\mathcal{H}_+(t)$ by (3). Hence, by continuity, this mapping can be extended to the closure of $P(D)(\mathcal{D})^m$ in $\mathcal{H}_+(t)$, and by zero to the orthogonal complement of the closure of $P(D)(\mathcal{D})^m$ in $\mathcal{H}_+(t)$.

Now, let G be the continuous linear mapping $\mathcal{H}_+(t) \to \mathcal{H}_+(t)$ defined above, and let G^* be a dual operator of G.

$$\mathcal{H}_-(t) \ni {}^\forall f, \qquad \text{for every } \varphi \in (\mathcal{D})^m,$$
$$\langle f, \varphi \rangle = \langle f, GP(D)\varphi \rangle = \langle G^* f, P(D)\varphi \rangle$$

Put $U = G^* f \in \mathcal{H}_-(t)$.

$$\langle f, \varphi \rangle = \langle U, P(D)\varphi \rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} \int u_i(X) P_{ij}(D) \varphi_j(X) dX$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \int P_{ij}(-D) u_i(X) \cdot \varphi_j(X) dX = \langle {}^t P(-D) U, \varphi \rangle$$

Hence ${}^tP(-D)U = f$ in the distributional sense. Since $P(D)$ is arbitrary except condition (C), and condition (C) is kept for the exchanging $P(D)$ and ${}^tP(-D)$, therefore the following theorem holds.

Theorem 2. *Let $P(X)$ be the square matrix satisfying condition* (C). *Then, for every vector-valued function $U \in \mathcal{H}_-(t)$, there exists a vector-valued function $U \in \mathcal{H}_-(t)$ such that $P(D)U = f$ in the distributional sense.*

On the regularity. Let $\mathcal{H}^{(k)}_-(t)$ be the linear space of $f(X)$ such that the distributional derivative $D^\alpha f(X) = (D^\alpha f_1, \cdots, D^\alpha f_n) \in \mathcal{H}_-(t)$ for $|\alpha| \leqq k$, with the inner product $(f, g)_{k,-,t} = \sum_{|\alpha| \leqq k} (D^\alpha f, D^\alpha g)_{-,t}$, and the norm $\|f\|^{(k)}_{-,t} = \sqrt{(f, f)_{k,-,t}}$. Then $\mathcal{H}^{(k)}_-(t)$ is a Hilbert space. Similarly, $\mathcal{H}^{(k)}_+ t = \{f\,;\, D^\alpha f \in \mathcal{H}_+(t) \text{ for } |\alpha| \leqq k\}$ is a Hilbert space with the inner product $(f, g)_{k,+,t} = \sum_{|\alpha| \leqq k} (D^\alpha f, D^\alpha g)_{+,t}$. Now, the bilinear form on $\mathcal{H}^{(k)}_+(t) X \mathcal{H}^{(k)}_-(t)$ is defined by $\langle f, g \rangle_k = \sum_{|\alpha| \leqq k} \langle D^\alpha f, D^\alpha g \rangle$. Then this bilinear form $\langle\ ,\ \rangle_k$ is continuous.

The next estimate follows immediately from estimate (3):

$$\|\varphi\|^{(k)}_{+,t} \leqq C\,\|P(D)\varphi\|^{(k)}_{+,t} \qquad \text{for every } \varphi \in (\mathcal{D})^m.$$

Hence, by above estimate, there exists a continuous linear mapping G from $\mathcal{H}^{(k)}_+(t)$ to $\mathcal{H}^{(k)}_+(t)$ such that $GP(D)\varphi = \varphi$ for every $\varphi \in (\mathcal{D})^m$ Then, by transposition, G defined a continuous linear mapping ${}^tG : \mathcal{H}^{(k)}_-(t) \to \mathcal{H}^{(k)}_-(t)$.

Therefore, the following theorem holds alike in above Theorem 2.

Theorem 3. *Let k be any positive integer, and let $P(X)$ be the square matrix satisfying condition* (C). *Then, for every $f \in \mathcal{H}^{(k)}_-(t)$, there exists a solution $U \in \mathcal{H}^{(k)}_-(t)$ such that $P(D)U = f$ in the sense of distribution.*