119. A Path Space and the Propagation of Chaos for a Boltzmann's Gas Model

By Tadashi UENO
Department of Mathematics, Faculty of General Education, University of Tokyo

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In [3], we considered a model with infinite number of interacting particles. A gas model corresponding to a *spatially homogeneous Boltzmann equation with bounded scattering cross section* can be discussed in the frame work, but *not* the gas of hard spheres.[1)]

Here, we construct a path space which describes, to some extent, the motion in the model, especially the way of interactions between particles. Next, we formulate a natural version of the *propagation of*

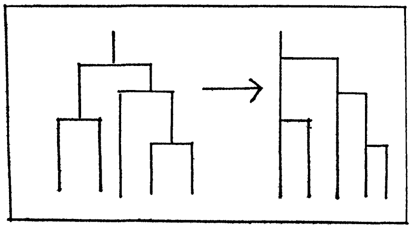

Fig. 1

chaos, which Kac [1] discovered. The version needs no approximation process with respect to the number of particles.

We use the notations and definitions in [3], but rewrite the figure of branches as in Fig. 1.[2)]

1. Following two lemmas are fundamental.

Lemma 1. *For x in R^∞ and $s \leqslant t$,*

$$\sum_{b \in T} P(s, b(x), t, R) \leqslant 1. \tag{2}$$

In case $q(t, x)$ is bounded,

$$\sum_{b \in T} P(s, b(x), t, R) \equiv 1, \qquad x \in R^\infty. \tag{3}$$

Here, $b(x)$ for $x=(x_1, x_2, \cdots)$ in R^∞ denotes $b((x_1, x_2, \cdots, x_{\#(b)}))$.

Lemma 2. *For a branch b, $x=(x_1, \cdots x_{\#(b)})$, $E \in \mathscr{B}(R)$ and $s \leqslant t \leqslant u$,*

$$P(s, b(x), u, E) = \sum_{b' \leqslant b} \int_{R^{\#(b')}} \prod_{b_i \in b/b'} P(s, b_i(x_i), t, dy_i) P(t, b'(y), u, E). \tag{4}$$

1) The space in which the particles move here or in [3] is the velocity space in the original gas model. The reader can consult McKean [2] for a more realistic description and related problems.

2) We restrict ourselves to binary interactions as in II of [3].

Lemma 1 is proved by an induction with respect to length $l(b)$. The proof of Lemma 2 has been outlined in [3].

2. *Transition probability.* Let $\bar{R}$ be $R\cup\{\partial\}$ with ∂ as an isolated point. For x in $R\times\bar{R}^{\infty}$ or in $R\times\bar{R}^{n-1}$, $\sharp(x)$ is the number of the components of x which are not ∂. When $\sharp(x)\geqslant\sharp(b)$ and $x_1, x_{n_1+1}, x_{n_1+n_2+1}, \cdots$ are the component of x which are not ∂, $b(x)$ denotes $b((x_1, x_{n_1+1}, \cdots, x_{n_1+\cdots+n_{\sharp(b)-1}+1}))$.

If $\sharp(x)=\infty$ for $x\in R\times\bar{R}^{\infty}$, we define for $E\in\mathscr{B}(R\times\bar{R}^{n-1})$ of type

$$E=E_1\times\{\partial\}^{n_1-1}\times E_2\times\{\partial\}^{n_2-1}\times\cdots\times E_k\times\{\partial\}^{n_k-1},\quad n_1+\cdots+n_k=n,\ E_i\in\mathscr{B}(R),\ 1\leqslant i\leqslant k, \tag{5}$$

$$\begin{aligned} P^{(n)}(s,x,t,E)=&\prod_{i=1}^{k-1}\sum_{\sharp(b_i)=\sharp(x_i)}P(s,b_i(x_i),t,E_i)\\ &\times\sum_{\sharp(b_k)\geqslant\sharp(x_k)}P(s,b_k(x'),t,E_k),\\ &\qquad\text{if } x_{n_1+1}, x_{n_1+n_2+1},\cdots,x_{n_1+\cdots+n_{k-1}+1} \text{ are not } \partial\\ =&\,0,\quad\text{if otherwise,}\end{aligned} \tag{6}$$

where $x_1=(x_1,\cdots,x_{n_1}),\cdots,x_k=(x_{n_1+\cdots+n_{k-1}+1},\cdots,x_n)$ and $x'=(x_{n_1+\cdots+n_{k-1}+1},\cdots)\in R\times\bar{R}^{\infty}$. This is extended uniquely to a measure $P^{(n)}(s,x,t,E)$ on $\mathscr{B}(R\times\bar{R}^{n-1})$.

Then, we can prove easily, by Lemma 1,

Theorem 1. *$P^{(n)}(s,x,t,E)$ is a substochastic measure. In case* (3) *holds, it is a probability measure and satisfies a consistency condition*:

$$P^{(n+1)}(s,x,t,E\times\bar{R})=P^{(n)}(s,x,t,E),\qquad\text{for } E\in\mathscr{B}(R\times\bar{R}^{n-1}). \tag{7}$$

Hence, under the condition (3), there is a unique probability measure $P(s,x,t,E)$ on the topological Borel field $\mathscr{B}(R\times\bar{R}^{\infty})$ of $R\times\bar{R}^{\infty}$ such that

$$P(s,x,t,E\times\bar{R}^{\infty})=P^{(n)}(s,x,t,E),\ E\in\mathscr{B}(R\times\bar{R}^{n-1}) \tag{8}$$

by the Kolmogorov extension theorem. Thus, *we always assume* (3) *from now on.*

For $x\in R\times\bar{R}^{n-1}$ and $E\in\mathscr{B}(R\times\bar{R}^{n-1})$ of type (6), we put

$$\begin{aligned} P_n(s,x,t,E)=&\prod_{i=1}^{k}\sum_{\sharp(b_i)=\sharp(x_i)}P(s,b_i(x_i),t,E_i),\\ &\qquad\text{if } x_{n_1+1},\cdots,x_{n_1+\cdots+n_{k-1}+1} \text{ are in } R,\\ =&\,0,\qquad\text{if otherwise.}\end{aligned} \tag{9}$$

This is also extended to a substochastic measure on $\mathscr{B}(R\times\bar{R}^{n-1})$ uniquely.

In case $\sharp(x)<\infty$ for $x\in R\times\bar{R}^{\infty}$, let n be the last coordinate number such that the corresponding component x is in R. We put

$$P(s,x,t,E_1\times E_2)=P_n(s,x_1,t,E_1)\times\delta_{\{\partial^\infty\}}(E_2), \tag{10}$$

for $E_1\in\mathscr{B}(R\times\bar{R}^{n-1})$ and $E_2\in\mathscr{B}(R\times\bar{R}^{\infty})$, where $x_1=(x_1,\cdots,x_n)$. Then, we define $P(s,x,t,E)$, in case $\sharp(x)<\infty$, as the unique extension of (10) on $\mathscr{B}(R\times\bar{R}^{\infty})$.

Let S be the set of all x such that $\sharp(x)=\infty$. Then, it can be proved that $P(s, x, t, S)=1$ or 0, according as $\sharp(x)=\infty$ or not.

Theorem 2. *$P(s, x, t, E)$ satisfies the Chapman-Kolmogorov equation:*

$$\begin{aligned}(11)\qquad &\int_{R\times\bar{R}^\infty} P(s, x, t, dy)P(t, y, u, E) \\ &=P(s, x, u, E), \qquad x\in R\times\bar{R}^\infty, E\in\mathcal{B}(R\times\bar{R}^\infty).\end{aligned}$$

This is proved by combining the following (12)–(14), where the essential part of (12) and (13) is reduced to Lemma 2.

$$(12)\quad \int_{R\times\bar{R}^{n-1}} P_n(s, x, t, dy) \sum_{\sharp(b)=\sharp(y)} P_n(t, b(y), u, E)=\sum_{\sharp(b)=\sharp(x)} P_n(s, b(x), u, E).$$

$$\begin{aligned}(13)\qquad &\int_{R\times\bar{R}^\infty} P(s, x, t, dy) \sum_{\sharp(b)\geqslant\sharp(y|n)} P(t, b(y), u, E) \\ &=\sum_{\sharp(b)\geqslant\sharp(x|n)} P(s, b(x), u, E),\end{aligned}$$

where $x|_n=(x_1, \cdots, x_n)$ and $y|_n=(y_1, \cdots, y_n)$ for $x=(x_1, \cdots)$ and $y=(y_1, \cdots)$ in $R\times\bar{R}^\infty$. For bounded measurable functions f_1 and f_2 on $R\times\bar{R}^{n-1}$ and $R\times\bar{R}^\infty$,

$$\begin{aligned}(14)\qquad &\int_{R\times\bar{R}^{n-1}\times R\times\bar{R}^\infty} P(s, x, t, dy)f_1(y_1)f_2(y_2) \\ &=\int_{R\times\bar{R}^{n-1}} P_n(s, x_1, t, dy_1)f_1(y_1)\times\int_{R\times\bar{R}^\infty} P(s, x_2, t, dy_2)f_2(y_2),\end{aligned}$$

where $x_1=(x_1, \cdots, x_n)$, $x_2=(x_{n+1}, \cdots)$, $y_1=(y_1, \cdots, y_n)$ and $y_2=(y_{n+1}, \cdots)$.

The minimal solution $P^{(f)}(s, x, t, E)$ of (1) in [3]–II and $P(s, x, t, E)$ is related as in

Theorem 3. *For each probability measure f on $(R, \mathcal{B}(R))$*

$$(15)\qquad P^{(f)}(s, x, t, E)=\int_{R^\infty} \prod_{i=2}^{\infty} f(dx_i)P(s, x, t, E\times\bar{R}^\infty).$$

3. *Path space.* Now, let I be the time interval and let Ω be the set of all functions on I which take values on $R\times\bar{R}^\infty$, or simply, $\Omega=(R\times\bar{R}^\infty)^I$. Let $\mathcal{B}$ be the Borel field generated by all cylinder sets of Ω, and let $x_t(\omega)=(x_t^{(1)}(\omega), x_t^{(2)}(\omega), \cdots)$ be the coordinate function for $\omega\in\Omega$ at time $t\in I$. Then, by Theorem 2 and the Kolmogorov extension theorem, there is a unique probability measure $P_{(t,x)}(\cdot)$ on $(\Omega, \mathcal{B})$ for each (t, x), such that

$$\begin{aligned}&P_{(t,x)}(\{\omega\in\Omega \mid x_{t_1}(\omega)\in A_1, \cdots, x_{t_k}(\omega)\in A_k\}) \\ (16)\qquad &=\int_{A_1} P(t, x, t_1, dy_1)\int_{A_2}\cdots\int_{A_{k-1}} P(t_{k-2}, y_{k-2}, t_{k-1}, dy_{k-1}) \\ &\qquad\cdot P(t_{k-1}, y_{k-1}, t_k, A_k),\end{aligned}$$

for $A_1, \cdots, A_k\in\mathcal{B}(R\times\bar{R}^\infty)$ and $t\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_k$.

Thus, the sytem $(\Omega, \mathcal{B}, P_{(t,x)}(\cdot), R\times\bar{R}^\infty, x_t)$ defines a Markov process with transition probability $P(s, x, t, E)$.

The idea of the path space is in Fig. 2. Each particle in R moves independently with others until a natural jump time of its own comes, or it suffers an interaction with another particle in R, whose coordinate

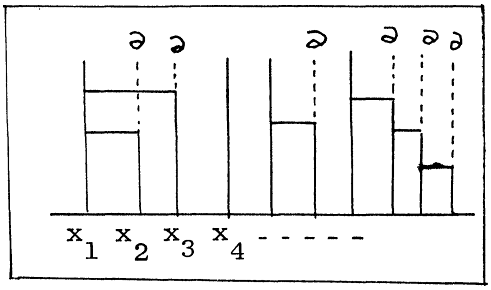

Fig. 2

is the nearest left hand neighbour of its coordinate.

If a natural jump time comes first, it jumps interacting with the particle in R with the nearest coordinate at the right hand, and then continues the motion in R. Here, the jumping measure $\pi(x' \mid t, x, E)$ depends on the position x' of the right hand particle in R.

On the other hand, if the particle suffers an interaction with the particle at the left hand side, it jumps instantly into ∂, while every particle at ∂ stands still and never interacts with others.

In a customary expression, the particles at point ∂ may be called *killed*. But, in our model, they should be considered as *forgotten* or *ignored* from the point of view of the particles remained in R at the left side, just because they will never interact with the remaining particles again. Thus, they stand still at ∂ only in appearance. This can be explained in terms of the original gas model, where the gas is so dilute that the second or later interactions between the same particles can (or should) be ignored.

4. *Propagation of chaos.* Although the particles in a Boltzmann's gas model are interacting, there is a phenomenon called the propagation of chaos. If all particles start independently with a same distribution f at time s, then, they are distributed independently with a common distribution at any later time t, where the distribution of each particle is the solution of the Boltzmann equation at time t with initial data f at time s. In a word, the chaos at the initial time s propagates for later times. Mark Kac [1] discovered this, and formulated and proved it for his one-dimensional model of a Maxwellion gas.

Here, we formulate a version of this as follows. Fix a time $\bar{t}>s$, and let $N_1(\omega) \equiv 1$, $N_2(\omega) = \inf\{n \geqslant 1 \mid x_{\bar{t}}^{(N_1+n)}(\omega) \in R\}$, $\cdots$, $N_m(\omega) = \inf\{n \geqslant 1 \mid x_{\bar{t}}^{(N_1+-+N_{m-1}+n)}(\omega) \in R\}$, and write $\tilde{x}_t^{(1)}(\omega) = x_t^{(N_1)}(\omega)$, $\tilde{x}_t^{(2)}(\omega) = x_t^{(N_1+N_2)}(\omega)$, $\cdots$, $x_t^{(m)}(u) = x_t^{(N_1+\cdots+N_m)}(\omega)$ for $t \in [s, \bar{t}]$. In a word, $\tilde{x}_t^{(i)}(\omega)$ is the i-th particle from the left which is remaining in R up to time $\bar{t}$.

Theorem 4. *For a probability measure f on $\mathscr{B}(R)$ and times $s<t_1 <\cdots<t_k=\bar{t}$,*

$$\begin{aligned}(17)\quad &\int_{R^\infty} f^\infty(d\boldsymbol{x}) P_{(s,\boldsymbol{x})}((\tilde{x}_{t_1}^{(1)}, \cdots, \tilde{x}_{t_k}^{(1)}) \in A_1, \cdots, (\tilde{x}_{t_1}^{(m)}, \cdots, \tilde{x}_{t_k}^{(m)}) \in A_m)) \\ &= \prod_{i=1}^{m} \int_{R^\infty} f^\infty(d\boldsymbol{x}) P_{(s,\boldsymbol{x})}((x_{t_1}^{(1)}, \cdots, x_{t_k}^{(1)}) \in A_i).\end{aligned}$$

When $A_i = A_{1,i} \times \cdots \times A_{k,i}, i=1,2,\cdots m$, *this is equal to*

$$\prod_{i=1}^{m} \int_{A_{1,i}} P_{s,t_1}^{(f)}(dx_1) \int_{A_{2,i}} P^{(f_{t_1})}(t_1, x_1, t_2, dx_2) \cdots \int_{A_{m-1,i}} P^{(f_{t_{m-2}})}(t_{m-2}, x_{m-2}, t_{m-1}, dx_{m-1}) P^{(f_{t_{m-1}})}(t_{m-1}, x_{m-1}, t_m, A_{m,i}).$$

Here, $A_1, \cdots, A_m$ *are in* $\mathcal{B}(R^k)$ *and* $f^\infty(d\boldsymbol{x})$ *is the infinite direct product of* $f(dx_i), i=1,2,\cdots.$ $f_{t_1} = \int_R f P_{s,t_1}^{(f)}$ and $f_{t_i} = \int_R f_{t_{i-1}} P_{t_{i-1},t_i}^{(f_{t_{i-1}})}$, $2 \le i \le m$.

This is a kind of strong Markov property with respect to the coordinate number n. The proof is reduced to (14).

The proof of the propositions will be published elsewhere.

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