117. Modules over Bounded Dedekind Prime Rings. II

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This paper is a continuation of [3]. Let D be an s-local domain which is a principal ideal ring. Then every right (left) ideal is an ideal and every ideal of D is a power of $J(D)$(see [2]). We put $J(D)=p_0D=Dp_0$. Then every non-unit $d\in D$ can be uniquely expressed as $d=p_0^k\varepsilon=\varepsilon'p_0^k$, where ε, ε' are units of D and k is an integer.

Let M be a D-module. An element x in M has *height* n if x is divisible by p_0^n but not by p_0^{n+1}; it has *infinite height* if it is divisible by p_0^n for every n. We write $h(x)$ for the height of x; thus $h(x)$ is a (non-negative) integer or the symbol ∞. Terminology and notation will be taken from [3].

Lemma 1. *Let D be an s-local domain which is a principal ideal ring, let M be a D-module and let S be a submodule with no elements of infinite height. Suppose that the elements of order $J(D)$ in S have the same height in S as in M. Then S is pure.*

Lemma 2. *Let D be an s-local domain which is a principal ideal ring and let M be a D-module. Suppose that all elements of order $J(D)$ in M have infinite height. Then M is divisible.*

An R-module is said to be *reduced* if it has no non-zero divisible submodules.

Theorem 1. *Let R be a bounded Dedekind prime ring and let P be a prime ideal of R. If M is a P-primary reduced R-module, then M possesses a direct summand which is isomorphic to eR/eP^n, where e is a uniform idempotent contained in R_P.*

By Theorem 1, we have

Theorem 2. *Let R be a bounded Dedekind prime ring. Then*

(i) *An finitely generated indecomposable R-module cannot be mixed and is not divisible, i.e., it is either torsion-free or torsion. In the former case, it is isomorphic to a uniform right ideal of R and in the latter case, it is isomorphic to eR/eP^n for some prime ideal P, where e is a uniform idempotent contained in R_P.*

(ii) *An indecomposable torsion R-module is either of type P^∞ or isomorphic to eR/eP^n for some prime ideal P, where e is a uniform idempotent contained in R_P.*

Lemma 3. *Let D be an s-local ring with $J(D)=p_0D$ which is a principal ideal domain. Let M be a D-module, let H be a pure submodule* *and let x be an element of order $J(D)$ not in H. Suppose that $h(x) = n < \infty$ and suppose further that $h(x+a) \leqq h(x)$ for every a in H with $O(a) = J(D)$. If K is the cyclic submodule generated by y with $x = yp_0^n$ and if $L = H + K$, then L is the direct sum of H and K, and L is pure again.*

A D-module M is of *bounded height* if there exists a constant k such that $h(x) \leqq k$ for all x in M. A set $\{x_i\}$ of elements of M is *pure independent* if the sum $\sum x_i D$ is direct and pure in M.

Lemma 4. *Let D be an s-local ring with $J = p_0 D$ which is a principal ideal domain. Let M be a D-module and let A be the submodule of elements x satisfying $O(x) = J$. Suppose that B, C are submodules of A, with $C \subseteq B \subseteq A$, and that B is of bounded height. If $\{x_i\}$ is a pure independent set satisfying $\sum \oplus x_i D \cap A = C$, then $\{x_i\}$ can be enlarged on a pure independent set $\{y_j\}$ satisfying $\sum \oplus y_j D \cap A = B$.*

Lemma 5. *Let P be a prime ideal of a bounded Dedekind prime ring R and let $R_P = (D)_k$, where $D = e_{11} R_P e_{11}$ and e_{11} is the matrix with 1 in the (1,1) position and zeros elsewhere. If M is a P-primary R-module, then M is a direct sum of cyclic R-modules if and only if Me_{11} is a direct sum of cyclic D-modules.*

Lemma 6. *With the same R, P, D and M as in Lemma 5, suppose that A is the D-submodule of elements x of Me_{11} satisfying $O(x) = J(D)$. Then a necessary and sufficient condition for M to be a direct sum of cyclic R-modules is that A be the union of an ascending sequence of D-submodules with bounded height.*

Now let M be a P-primary R-module and let x be a non-zero element of M. Then x has *height* n if $x \in MP^n$ and $x \notin MP^{n+1}$, it has *infinite height* if $x \in MP^n$ for every n.

From Lemmas 3, 4, 5 and 6 we have

Theorem 3. *Let P be a prime ideal of a bounded Dedekind prime ring R and let M be a P-primary R-module. Suppose that A is the submodule of elements x of M satisfying $xP = O$. Then a necessary and sufficient condition for M to be a direct sum of cyclic R-modules is that A be the union of an ascending sequence of submodules with bounded height.*

Corollary. *Let R be a bounded Dedekind prime ring and let M be a countable primary R-module with no elements of infinite height. Then M is a direct sum of cyclic R-modules.*

From Theorem 3, we have

Theorem 4. *Let R be a bounded Dedekind prime ring and let M be a primary R-module which is a direct sum of cyclic R-modules. Then any submodule N of M is a direct sum of cyclic R-modules.*

Theorem 5. *Let R be a bounded Dedekind prime ring and let M*

be a decomposable R-module. Then any submodule of M is decomposable.

Let M be an R-module. We call $O(M)=\{r\in R\mid Mr=0\}$ an *order ideal* of M. If M is an n-dimensional in the sense of Goldie, then we write $n=\dim M$.

Now, let M be a finitely generated R-module. Then M is a direct sum of uniform right ideals and uniform cyclic R-modules by Theorem 1 of [3] and Theorem 1. Thus we have

Theorem 6. *Let R be a bounded Dedekind prime ring and let M be a finitely generated R-module. Then for a decomposition of M into the direct sum of uniform right ideals and uniform cyclic R-modules, suppose that:*

(i) *the number of direct summands of uniform right ideals is r,*

(ii) *the number of P-primary cyclic summands for a given prime ideal P is k_p, where $k_p\geqq 0$, and that the orders of these summands are*

$$P^{\alpha_{p1}}, P^{\alpha_{p2}}, \cdots, P^{\alpha_{pk_p}},$$

where

$$\alpha_{p1}\geqq\alpha_{p2}\geqq\cdots\geqq\alpha_{pk_p}.$$

For a decomposition of any submodule N of M into the direct sum of uniform right ideals and uniform cyclic R-modules, suppose that:

(i) *the number of direct summands of uniform right ideals is s,*

(ii) *the number of P-primary cyclic summands for a given prime ideal P is l_p, where $l_p\geqq 0$, and that the orders of these summands are*

$$P^{\beta_{p1}}, P^{\beta_{p2}}, \cdots, P^{\beta_{pl_p}},$$

where

$$\beta_{p1}\geqq\beta_{p2}\geqq\cdots\geqq\beta_{pl_p}.$$

Then

(a) $s\leqq r$

(b) $l_p\leqq k_p$ *for each prime ideal P.*

(c) $\beta_{pi}\leqq\alpha_{pi}$ $(i=1, 2, \cdots, l_p)$

(d) $r+\sum k_p=\dim M$ *and* $s+\sum l_p=\dim N$.

From Theorem 1 and Theorem 1 of [1], we have

Theorem 7. *Let P be a prime ideal of a bounded Dedekind prime ring R and let M be a P-primary R-module. If M is decomposable, then M is a direct sum of uniform cyclic R-modules and the cardinal number of uniform cyclic summands of a given order is an invariant of M.*

References

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