115. *On a Theorem of K. Baumgartner*

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Throughout, A will represent a ring with 1, B a unital subring of A which is Artinian semi-simple, and C, Z the centers of A, B, respectively. We shall use the following representation: $B=B_1\oplus\cdots\oplus B_n$, where B_j is an Artinian simple ring with the identity element f_j. Then, $Z_j=Zf_j$ is the center of B_j and $Z=Z_1\oplus\cdots\oplus Z_n$.

In what follows, we shall present a slight generalization of a theorem of K. Baumgartner [1] on division rings with finite centers and a sharpening of [1; Korollar 1].

Theorem 1. *Let Z be finite. If A is prime or Artinian semi-simple then the following conditions are equivalent:*

(1) *C is finite.*

(2) *The dimension $\dim_B B\cdot C$ of the completely reducible B-module $B\cdot C$ is finite.*

(3) *There exists an integer k such that $\dim_B B[c]<k$ for all $c\in C$.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3): Trivial.

(3)$\Rightarrow$(1): If c is an arbitrary element of C then there holds $\dim_{B_j} B_j[cf_j]\leqslant \dim_B B[c]<k$. Since $B_j\cdot Cf_j=B_j\otimes_{Z_j}Z_j\cdot Cf_j$, there holds $[Z_j[cf_j]:Z_j]=[B_j[cf_j]:B_j]<k/\dim_{B_j}B_j$. If P_j is the prime field of Z_j, we obtain $[P_j[cf_j]:P_j]\leqslant[Z_j[cf_j]:Z_j]\cdot[Z_j:P_j]<k\cdot[Z_j:P_j]/\dim_{B_j}B_j$. If A is a prime ring then it is easy to see that f_jAf_j is a prime ring and the center C_j of f_jAf_j is an integral domain. On the other hand, if A is Artinian semi-simple then $f_jAf_j(\cong\mathrm{Hom}\,({}_AAf_j, {}_AAf_j))$ is Artinian semi-simple and C_j is a finite direct sum of fields. Hence, in either case, the subring Cf_j of C_j is finite. It follows therefore the subring C of $Cf_1\oplus\cdots\oplus Cf_n$ is finite.

Theorem 2. *Let A be semi-prime and left finite over B. In order that Z be finite, it is necessary and sufficient that C be finite.*

Proof. As is well-known, the left Artinian semi-prime ring A is Artinian semi-simple: $A=A_1\oplus\cdots\oplus A_m$, where A_i is an Artinian simple ring with the identity element e_i. Now, by the validity of Theorem 1, it remains only to prove the sufficiency. Since Be_i is a homomorphic image of B, Be_i is evidently Artinian semi-simple and A_i is left finite over Be_i. Recalling here that $Z\subseteq Ze_1\oplus\cdots\oplus Ze_m$, we may restrict our attention to the case that A is simple. Then, there exists a system of matrix units $E=\{e_{pq} \mid p, q=1, \cdots, r\}$ such that $V_A(E)$(the centralizer of E in A) is a division ring D and each f_j is a sum of suitable e_{pp}'s. By making use of the representation $A=\sum_{p,q=1}^{r} De_{pq}$, we can easily see that the center of f_jAf_j coincides with the finite field Cf_j. Evidently, f_jAf_j is left finite over B_j. Therefore, we may assume further that B is simple. Then, by [2; Proposition 5.4 (b)], the rank of $V_A(B)$ over C does not exceed the left rank of A over B. Hence, the subfield Z of $V_A(B)$ is finite.

References

[1] K. Baumgartner: Schiefkörper mit endlichem Zentrum. Archiv der Math., **20**, 134–135 (1969).

[2] H. Tominaga and T. Nagahara: Galois Theory of Simple Rings. Okayama Mathematical Lectures (1970).