136. On Finite Groups whose Subgroups have Simple Core Factors*)

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If H is a subgroup of the finite group G then the core of H, denoted H_G, is $\bigcap_{x\in G} x^{-1}Hx$, the largest of the normal subgroups of G contained in H; the core factor of the subgroup H is H/H_G. G is called (see [1]) $\mathfrak{X}$-core (respectively, $\mathfrak{X}$-max-core) if all its subgroups (respectively, all its maximal subgroups) have core factors in the class $\mathfrak{X}$ of finite groups. $\mathfrak{X}$-max-core groups have been classified, for some $\mathfrak{X}$, in [1] and [4], but little is known about $\mathfrak{X}$-core groups. Of course, if $\mathfrak{X}=\{1\}$, $\mathfrak{X}$-core groups are precisely the Hamiltonian groups; the purpose of this paper is to give information about $\mathfrak{X}$-core groups close to Hamiltonian groups-that is, groups whose core factors are relatively uncomplicated.

Throughout this paper, all groups considered are finite. Unless otherwise specified, references and notation are drawn from [3]. Let $\mathfrak{S}, \mathfrak{C}$, and $\mathfrak{P}$ denote the classes of all groups which are simple, cyclic, and of prime-power order, respectively (including the trivial group). Then $\mathfrak{S}\cap\mathfrak{C}=\mathfrak{S}\cap\mathfrak{P}$ is the class of all groups of order a prime. We begin by showing that $\mathfrak{S}$-core groups are $\mathfrak{S}\cap\mathfrak{C}$-core groups.

(1) **Theorem.** *$\mathfrak{S}$-core groups are solvable.*

Proof. We recall from [1] that subgroups and homomorphic images of $\mathfrak{S}$-core groups are again $\mathfrak{S}$-core groups. Now let G be a minimal counterexample. If $1\neq N\lhd G, N\neq G$, then N and G/N are $\mathfrak{S}$-core groups and by induction must be solvable, making G solvable, a contradiction. Therefore G is simple. But then all subgroups of G have trivial core and hence must be simple too-even the Sylow subgroups. This means G has only cyclic Sylow subgroups and so by a theorem of Hölder (p. 420, [3]) is solvable.

(2) **Corollary.** *If G is an $\mathfrak{S}$-core groups and $H\leq G$ then H_G has index at most a prime in H.*

(3) **Corollary.** *If G is an $\mathfrak{S}$-core group then $F(G)$, the Fitting subgroup, has index at most a prime in G.*

Proof. This follows directly from Proposition (7) of [1].

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Are $\mathfrak{S}$-max-core groups also solvable? This reduces to:

(4) Conjecture. There are no simple groups all of whose maximal subgroups are simple.

For, if this conjecture holds, and G is an $\mathfrak{S}$-max-core group with M a maximal subgroup, then if $M_G \neq 1$, by induction M/M_G in G/M_G is solvable; while if $M_G = 1$, then let $1 \neq N \lhd G, N \neq G$ (possible if the conjecture is validated) and consider $M \cap N$: because $M \cap N \lhd M$ and $M = M/M_G \in \mathfrak{S}$ then $M \cap N = 1$ and $G = MN$; consequently $M = G/N$ is solvable. Therefore G is $\mathfrak{P}$-max-core and hence solvable by (3) of [1]. To illustrate the difficulty with the conjecture we note that in A_7, the the alternating group on seven letters, the Sylow subgroups and the centralizers of involutions are contained in simple maximal subgroups (see [2]).

A group G is called an A-group whenever G is a solvable group all of whose Sylow subgroups are abelian. If we let $Z_\infty(G)$ denote the hypercenter of G, our main result may be stated as follows:

(5) Theorem. *If G is a $\mathfrak{C} \cap \mathfrak{P}$-core group then $G/Z_\infty(G)$ is an A-group.*

We require

(6) Lemma. *Let G be a $\mathfrak{C} \cap \mathfrak{P}$-core group with $Z(G)=1$. Then G has a p-Sylow subgroup P satisfying*:

$G = F(G)P, P \cap F(G) = 1, P$ *cyclic, and* $N_G(P) = P$.

Proof. G is solvable by (3) of [1] and hence, by (7) of [1], $G/F(G)$ is cyclic of order a power of some prime p. Let P be a p-Sylow subgroup of G; then $G = F(G)P$. If $F(G) \cap P \neq 1$, then there is some element $1 \neq x \in F(G) \cap Z(P)$ (p. 301, [3]). But $F(G)$ is nilpotent and $P \cap F(G)$ is a p-Sylow subgroup of $F(G)$; hence $x \in Z(F(G))$. It follows that $x \in Z(PF(G)) = Z(G) = 1$. Therefore $P \cap F(G) = 1$ and $P = G/F(G) \in \mathfrak{C} \cap \mathfrak{P}$.

Now let $x \in N_G(P)$, x of prime order $q \neq p$, and let $H = \langle x, P \rangle = \langle x \rangle P$. Now if H_G has a non-trivial p-Sylow subgroup T, then $T = H_G \cap P \lhd H_G$. It follows that $T \lhd G$ so $T \subseteq F(G)$, a contradiction. Because G is a $\mathfrak{C} \cap \mathfrak{P}$-core group, then $H_G = \langle x \rangle$. But then $H = \langle x \rangle \times P$ so $x \in C_G(P)$, while $x \in Z(F(G))$ since $\langle x \rangle$ is a minimal normal subgroup of G (p. 277, [3]). Therefore $x \in Z(G) = 1$, and we are forced to conclude that $N_G(P) = P$.

Proof of (5). Let G be a minimal counterexample. Note that if T is a proper subgroup or factor group of G then T is $\mathfrak{C} \cap \mathfrak{P}$-core group and so $T/Z_\infty(T)$ is an A-group, by induction. Therefore, if $Z_\infty(G) \neq 1$, then $G/Z_\infty(G)$ is an A-group, for it has trivial center. It follows that $Z_\infty(G) = Z(G) = 1$, and hence by the lemma G has a cyclic p-Sylow subgroup P satisfying: $G = F(G)P, F(G) \cap P = 1, N_G(P) = P$. For simplicity, denote $F(G)$ by F.

We show next that G has a monolith K (that is, a unique minimal normal subgroup). Let N_1 and N_2 be minimal normal subgroups of G, $N_1 \neq N_2$ so $N_1 \cap N_2 = 1$. Define $T_i \geq N_i$ by $T_i/N_i = Z_\infty(G/N_i) = Z_{n_i}(G/N_i)$ $(i=1,2)$ and put $n = \max\{n_1, n_2\}$. Then $T_i = \{g \in G \mid [g, x_1, x_2, \cdots, x_n] \in N_i$, for all $x_1, x_2, \cdots, x_n \in G\}$ so $T_1 \cap T_2 = \{g \in G \mid [g, x_1, \cdots, x_n] \in N_1 \cap N_2 = 1$, for all $x_1, \cdots, x_n \in G\} \leq Z_\infty(G) = 1$. Now by induction, $G/T_i \simeq (G/N_i)/Z_\infty(G/N_i)$ is an A-group, and hence $G = G/(T_1 \cap T_2)$ must be an A-group, a contradiction. Therefore G has a monolith K. Since the Sylow subgroups of F are normal in G, F must have prime power order then, say $|F| = q^n$, and $K \leq F$.

If $1 \neq N \lneqq F$ and $N \lhd G$ then we can show that N is abelian and G/N is an A-group. We begin by noting that since $N_G(P) = P = G/F$ then (p. 737, [3]) P and its conjugates are the system normalizers and Carter subgroups of G. This is also true for P in NP and for NP/N in G/N (p. 737, [3]). Therefore $Z_\infty(NP) \leq P$ and $Z_\infty(G/N) \leq NP/N$ (p. 729, [3]). It follows that the q-Sylow subgroup at $NP/Z_\infty(NP)$ is isomorphic to N, while that of $(G/N)/Z_\infty(G/N)$ is isomorphic to F/N. By induction $NP/Z_\infty(NP)$ and $(G/N)/Z_\infty(G/N)$ are A-groups and hence N and F/N are abelian. Because F/K is abelian but F is not, $K = F'$.

In addition, we can show that every abelian subgroup of $F(G)$ has at most two generators. By considering the subgroup generated by the elements of order q, it suffices to show that $F(G)$ contains no elementary abelian subgroups of order greater than q^2. Let H be elementary abelian of order q^3. Because $H/H_G \in \mathfrak{C} \cap \mathfrak{P}$ then H_G has order q^2 or q^3 and $K \leq H_G$ (since G is monolithic). If $|K| = q^3$ then let $T \lneqq K$ have order p^2. Again $T/T_G \in \mathfrak{C} \cap \mathfrak{P}$ so $T_G \neq 1$ and we must have $K \leq T_G$, a contradiction. [This proves that in $G, |K| \leq q^2$; for, as a minimal normal subgroup of the solvable group G, K must be elementary abelian.] Now suppose $|K| = q^2$ and let $x \in K$ and $y \in H - K$. Then $\langle x, y \rangle = T$ is elementary abelian of order p^2, so $T_G \neq 1$. But $K \not\leq T$, contradicting the fact that K was the monolith of G. Finally, when $|K| = p$, take T to be the complement of K in H; again, $T_G \neq 1$, a contradiction.

At this point we could appeal to some rather deep results of Blackburn and of Thompson on the characterization of groups of prime power order whose abelian normal subgroups are at most two generator (see pp. 343–346, [3]), but elementary methods suffice. First, suppose q is odd. Because $F' = K, F$ has nilpotent class 2, so F is regular (p. 322, [3]). Consider $\Omega = \Omega(F) = \langle x \in F \mid x^q = 1 \rangle$. Either $\Omega = F$ or Ω, as a normal subgroup of G, properly contained in F, is abelian with at most two generators. In this latter case $|\Omega| \leq q^2$; but if $|\Omega| = q, F$ must be cyclic (p, 310, [3]), so $|\Omega| = q^2$ and F is metacyclic (p. 337, [3]). It follows that K is cyclic and so of order q. Suppose H is any one of the other q subgroups of order q of Ω. If $N_G(H)\cap P=\bar{P}\gneq 1$ then consider $H\bar{P}$: $(H\bar{P})_G\neq 1$ so $K\leq H\bar{P}$, contradicting that $H\lhd H\bar{P}$ is the q-Sylow subgroup of $H\bar{P}$. Therefore P and its non-trivial subgroups act as permutation groups on the subgroups of order q of Ω, fixing only K, from which we deduce $p\mid q$, a contradiction.

Therefore $F=\Omega$, and by regularity must be of exponent q. If H is maximal in F then $\Phi(F)\leq H_G$ so $|H:H_G|\leq q$; furthermore H_G is abelian with at most two generators, and has exponent q, so $|H_G|\leq q^2$. Hence $|F|\leq q^4$. Now the group of order q^4, exponent q and nilpotent class 2, has derived group of order q and center of order q^2, and so the argument of the above paragraph, with the center in place of Ω, shows that F cannot be this group. Hence $|F|=q^3, |K|=q$. Again we can use the above argument to deduce that no subgroup (of F) of order q^2 is normalized by any non-trivial subgroup of P. In this way P acts as a permutation group on the $q+1$ subgroups of order q^2, the stabilizer of each point being trivial. Hence $|P|$ divides $q+1$. On the other hand $|K|=q, K\lhd G, Z(G)=1$, and so $|N_G(K)/C_G(K)|$ is a non-trivial divisor of $q-1$, the order of the automorphism group of K. Therefore p divides $q-1$, as well as $q+1$. Consequently $p=2$. Let $x^2=1, 1\neq x\in P$. Then, as we have remarked, x moves every subgroup of order q^2, so x cannot fix any subgroup of order q of F/K, or in turn any non-trivial element of F/K. But the only fixed-point-free automorphism of order 2 is the inverting automorphism (p. 506, [3]) which fixes all subgroups, a contradiction.

We are forced to conclude that $q=2$. If $|K|=q=2$ then $K\leq Z(G)=1$, so we must have $|K|=4$. Let $H=\langle x_1, x_2, \Phi(F)\rangle$ be any subgroup of F satisfying $|H:\Phi(F)|=4$, which is possible since F is not cyclic. Now $H_G\geq\Phi(F)$ since $\Phi(F)\lhd G$, but $H_G\neq\Phi(F)$ because $H/\Phi(F)$ is not cyclic. And if $|H:H_G|=2$, say $H_G=\langle x_1, \Phi(F)\rangle$, then $H_G/\Phi(F)$ is a normal subgroup of order 2 of $G/\Phi(F)$, contradicting that $Z_\infty(G/\Phi(F))\leq P\Phi(F)/\Phi(F)$ (derived in proving $F'=K$). Therefore $H=H_G$; if $H\neq F$ then F has an element x_3 with $L=\langle x_1, x_3, \Phi(F)\rangle\neq H, |L:\Phi(F)|=4$ and so $L\lhd G$. But then $\langle x_1, \Phi(F)\rangle=H\cap L\lhd G$, which we showed above was a contradiction. It follows that $|F:\Phi(F)|=4$ and consequently $F'=K$ must be cyclic (p. 258, [3]), a contradiction.

This completes the proof.

Theorem (5) does not generalize to $\mathfrak{C}$-core groups: for example, the group $G=\langle a, b\mid a^{49}=b^{21}=1, a^b=a^{-5}\rangle$, which is the unique subgroup of index 2 in the holomorph of the cyclic group of order 49, has $Z(G)=Z_\infty(G)=1$ and both G/G' and G' are cyclic, so G is $\mathfrak{C}$-core, but G has a non-abelian Sylow subgroup.*)

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The dihedral groups provide an elementary example of non-Hamiltonian $\mathfrak{S}$-core groups. Further properties of $\mathfrak{C} \cap \mathfrak{P}$-core groups, and in particular $\mathfrak{S}$-core groups will be explored in a later paper.

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