135. A Note on Distributive Sublattices of a Lattice

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In [1], B. Jónsson gave a necessary and sufficient condition for a subset of a modular lattice to generate a distributive lattice. R. Balbes proved Jónsson's theorem without using Zorn's lemma in [2]. In [3], we gave a necessary and sufficient condition that the sublattice generated by a subset H of a lattice should be distributive. In this note we prove this theorem without using Zorn's lemma. And then the condition for the case of $H=\{x, y, z\}$ is expressed by seven lattice polynomial equations.

§1. The finite join of elements in H is called a $\cup$-element. The set of all $\cup$-elements is denoted by $H_\cup$ and dually the set of all $\cap$-elements by $H_\cap$. The finite join of elements in $H_\cap$ is called a $\cup\cap$-element. The set of all $\cup\cap$-elements is denoted by $H_{\cup\cap}$ and dually the set of all $\cap\cup$-elements by $H_{\cap\cup}$.

Two modular laws will be expressed by

μ: $(a\cap c)\cup(b\cap c)=((a\cap c)\cup b)\cap c$, and

μ^*: $(a\cup c)\cap(b\cup c)=((a\cup c)\cap b)\cup c$.

Four distributive laws will be expressed by

δ: $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$,

δ^*: $(a\cup c)\cap(b\cup c)=(a\cap b)\cup c$,

Δ: $\bigcup_{i=1}^m (x_i\cap y)=(\bigcup_{i=1}^m x_i)\cap y$, and

Δ^*: $\bigcap_{i=1}^m (x_i\cup y)=(\bigcap_{i=1}^m x_i)\cup y$.

Theorem 1. *Let H be a nonempty subset of a lattice L. In order for the sublattice of L generated by H to be distributive, it is necessary and sufficient that*

Δ holds for any $x_1, \cdots, x_m \in H$ and any $y\in H_\cap$,

μ holds for any $a\in H_\cap$ and any $b, c\in H_{\cup\cap}$, and

μ^* holds for any $b\in H_\cap$ and any $a, c\in H_{\cup\cap}$.

Proof. The modular laws used in the proof of [2] are only those laws mentioned above.

Corollary 2. *Let $\langle H\rangle$ be the sublattice generated by a nonempty subset H of a lattice. The following four statements are equivalent.*

(i) *$\langle H\rangle$ is distributive.*

(ii) *δ holds for any $a, b, c\in H_{\cup\cap}$.*

(iii) *Δ holds for any $x_1, \cdots, x_m\in H$ and any $y\in H_\cap$, and*
μ^* holds for any $a, b, c\in H_{\cup\cap}$.

(iv) *Δ holds for any $x_1, \cdots, x_m \in H$ and any $y \in H_\cap$,*
μ holds for any $a \in H_\cap$ and any $b, c \in H_{\cup\cap}$, and
μ^* holds for any $b \in H_\cap$ and any $a, c \in H_{\cup\cap}$.

§2. Each of the following six nonselfdual distributive laws is called δ-law.

$\delta(1)$: $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.
$\delta(2)$: $y \cap (z \cup x) = (y \cap z) \cup (y \cap x)$.
$\delta(3)$: $z \cap (x \cup y) = (z \cap x) \cup (z \cap y)$.
$\delta^*(1)$: $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.
$\delta^*(2)$: $y \cup (z \cap x) = (y \cup z) \cap (y \cup x)$.
$\delta^*(3)$: $z \cup (x \cap y) = (z \cup x) \cap (z \cup y)$.

Each of the following twelve nonselfdual modular laws is called μ-law.

$\mu(1)$: $x \cap (y \cup (x \cap z)) = (x \cap y) \cup (x \cap z)$.
$\mu(1')$: $x \cap (z \cup (x \cap y)) = (x \cap z) \cup (x \cap y)$.
$\mu(2)$: $y \cap (z \cup (y \cap x)) = (y \cap z) \cup (y \cap x)$.
$\mu(2')$: $y \cap (x \cup (y \cap z)) = (y \cap x) \cup (y \cap z)$.
$\mu(3)$: $z \cap (x \cup (z \cap y)) = (z \cap x) \cup (z \cap y)$.
$\mu(3')$: $z \cap (y \cup (z \cap x)) = (z \cap y) \cup (z \cap x)$.
$\mu^*(1)$: $x \cup (y \cap (x \cup z)) = (x \cup y) \cap (x \cup z)$.
$\mu^*(1')$: $x \cup (z \cap (x \cup y)) = (x \cup z) \cap (x \cup y)$.
$\mu^*(2)$: $y \cup (z \cap (y \cup x)) = (y \cup z) \cap (y \cup x)$.
$\mu^*(2')$: $y \cup (x \cap (y \cup z)) = (y \cup x) \cap (y \cup z)$.
$\mu^*(3)$: $z \cup (x \cap (z \cup y)) = (z \cup x) \cap (z \cup y)$.
$\mu^*(3')$: $z \cup (y \cap (z \cup x)) = (z \cup y) \cap (z \cup x)$.

The following selfdual distributive law is called D-law.

D: $(y \cap z) \cup (z \cap x) \cup (x \cap y) = (y \cup z) \cap (z \cup x) \cap (x \cup y)$.

Each of the following three selfdual modular laws is called M-law.

M(1): $(y \cap z) \cup (x \cap (y \cup z)) = ((y \cap z) \cup x) \cap (y \cup z)$.
M(2): $(z \cap x) \cup (y \cap (z \cup x)) = ((z \cap x) \cup y) \cap (z \cup x)$.
M(3): $(x \cap y) \cup (z \cap (x \cup y)) = ((x \cap y) \cup z) \cap (x \cup y)$.

Lemma 3. *Let L be a lattice and $x, y, z \in L$. Assume seven lattice polynomial equations* $\delta(1)$, $\mu^*(3)$, $\mu(2)$, $\mu^*(1)$, $\mu(3)$, $\mu^*(2)$, *and* M(1). *Then all δ-laws, μ-laws, D-law and M-laws are asserted.*

Proof. $\delta(1)$ and $\mu^*(3)$ imply $\delta^*(3)$.

$(z \cup x) \cap (z \cup y)$
$= z \cup (x \cap (z \cup y)) \quad \cdots\cdots$ (By $\mu^*(3)$)
$= z \cup (x \cap z) \cup (x \cap y) \quad \cdots\cdots$ (By $\delta(1)$)
$= z \cup (x \cap y)$.

We shall denote this fact, as follows,

$$\delta(1) \xrightarrow{\mu^*(3)} \delta^*(3).$$

Using this notation, we shall have the following sequence of the proof.

$$\delta(1)\xrightarrow{\mu^*(3)}\delta^*(3)\xrightarrow{\mu(2)}\delta(2)\xrightarrow{\mu^*(1)}\delta^*(1)\xrightarrow{\mu(3)}\delta(3)\xrightarrow{\mu^*(2)}\delta^*(2).$$

Thus we have all δ-laws.

$\delta(1)$ and $\delta^*(2)$ imply $\mu(1)$.

$x\cap(y\cup(x\cap z))$

$=x\cap(y\cup x)\cap(y\cup z)$ $\cdots\cdots$ (By $\delta^*(2)$)

$=x\cap(y\cup z)$

$=(x\cap y)\cup(x\cap z)$ $\cdots\cdots$ (By $\delta(1)$)

Similarly all μ-laws are proved by using two δ-laws.

$\delta(1), \delta^*(1)$ and M(1) imply D.

$(y\cap z)\cup(z\cap x)\cup(x\cap y)$

$=(y\cap z)\cup(x\cap(y\cup z))$ $\cdots\cdots$ (By $\delta(1)$)

$=((y\cap z)\cup x)\cap(x\cup z)$ $\cdots\cdots$ (By M(1))

$=(y\cup z)\cap(z\cup x)\cap(x\cup y)$ $\cdots\cdots$ (By $\delta^*(1)$)

$\delta(2), \delta^*(2)$ and D imply M(2).

$(z\cap x)\cup(y\cap(z\cup x))$

$=(z\cap x)\cup(y\cap z)\cup(y\cap x)$ $\cdots\cdots$ (By $\delta(2)$)

$=(z\cup x)\cap(y\cup z)\cap(y\cup x)$ $\cdots\cdots$ (By D)

$=((z\cap x)\cup y)\cap(z\cup x)$ $\cdots\cdots$ (By $\delta^*(2)$)

Similarly M(3) is provable.

Theorem 4. *Suppose L is a lattice and $x, y, z\in L$. In order for the sublattice of L generated by the set $\{x, y, z\}$ to be distributive it is necessary and sufficient that*

$x\cap(y\cup z)=(x\cap y)\cup(x\cap z)$,

$z\cup(x\cap(z\cup y))=(z\cup x)\cap(z\cup y)$,

$y\cap(z\cup(y\cap x))=(y\cap z)\cup(y\cap x)$,

$x\cup(y\cap(x\cup z))=(x\cup y)\cap(x\cup z)$,

$z\cap(x\cup(z\cap y))=(z\cap x)\cup(z\cap y)$,

$y\cup(z\cap(y\cup x))=(y\cup z)\cap(y\cup x)$, *and*

$(y\cap z)\cup(x\cap(y\cup z))=((y\cap z)\cup x)\cap(y\cup z)$.

Proof. We can prove it tediously but easily by Theorem 1 and Lemma 3.

References

[1] B. Jónsson: Distributive sublattices of a modular lattice. Proc. Amer. Math. Soc., **6**, 682–688 (1955).

[2] R. Balbes: A note on distributive sublattices of a modular lattice. Fundamenta Mathematicae, **65**, 219–222 (1969).

[3] S. Tamura: On distributive sublattices of a lattice. Proc. Japan Acad., **47**, 442–446 (1971).