171. *A New Characterization of Real Analytic Functions*

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In [1], [2] we have shown that the local operators, the differential operators of infinite order in the theory of hyperfunctions, characterize the hyperfunctions in terms of measures. Here we present a theorem which shows that the local operators also characterize the real analytic functions in terms of continuous, functions. We mention the case of one variable. The case of several variables can be proved in the same way when we apply the result of [2] and use the standard defining functions of the hyperfunctions with compact supports of several variables. It will be published in a forthcoming paper [3].

Proposition. *Let $A_J(K)$ be the space of the real analytic functions on the compact interval K with its topology defined by the seminorms $\|u\|_J=\sup_{x\in K}|J(D)u(x)|$, where $J(D)$ runs over the local operators with constant coefficients* (see [1]). *Then $A_J(K)$ is sequentially complete.*

Proof. By Corollary 6 of [1] (and by a trivial consideration) we have the following domination relation between the topologies of $A(K)$:

$$\sigma(A(K), B[K]) \prec A_J(K) \prec A(K),$$

where $A(K)$ denotes its usual DFS topology, and $\sigma(A(K), B[K])$ the weak topology of $A(K)$ under pairing with its dual space $B[K]$ (the space of hyperfunctions with supports in K). Thus by the theorem of Mackey the bounded sets are common in all these spaces. Since $A(K)$ is DFS, its bounded sets are relatively compact. Now take a Cauchy sequence $\{f_n\}$ in $A_J(K)$. Then $\{f_n\}$ is obviously bounded in $A_J(K)$. Thus it is bounded in $A(K)$, hence relatively compact. Therefore there is a subsequence $\{f_{n'}\}$ which converges to some element $f\in A(K)$ in $A(K)$, hence in $A_J(K)$. Since $\{f_n\}$ is a Cauchy sequence, we have proved that $\{f_n\}$ converges to a real analytic function f in $A_J(K)$. q.e.d.

Theorem. *Let f be a hyperfunction on an open interval $I\subset R^1$. Assume that for any local operator J, $J(D)f$ is a continuous function in I. Then f is real analytic in this interval.*

Proof. Since we are concerned with a local property, we only have to prove that f is real analytic in the interior of any compact

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subinterval $K \subset I$. Take a defining function $\varphi \in \mathcal{O}(C^1 \setminus I)$ such that $f(x) = \varphi(x+i0) - \varphi(x-i0)$. Then by the definition of the operation of $J(D)$, $J(D)\varphi \in \mathcal{O}(C^1 \setminus I)$ is the defining function of $J(D)f$, namely $J(D)f(x) = J(D)\varphi(x+i0) - J(D)\varphi(x-i0)$. Moreover, since $J(D)f$ is continuous in I, the sequence of real analytic functions: $J(D)f_n(x) = J(D)\varphi(x+i/n) - J(D)\varphi(x-i/n)$ converges to $J(D)f(x)$ uniformly on K. (This follows easily from the classical theory of Poisson Kernels). Thus $\{f_n\}$ is a Cauchy sequence in $A_J(K)$. By Proposition, there exists some $g \in A(K)$ such that $\{f_n\}$ converges to g in $A_J(K)$. Thus $\{f_n\}$ converges to f and also to g in $C(K)$, the space of the continuous functions on K with uniform convergence topology. Since it is Hausdorff, we conclude that $f = g$, so that, f is real analytic in the interior of K. q.e.d.

The following is a special case of our theorem.

Corollary. *Let $f(z)$ be a holomorphic function in $V \cap H$, where $V \subset C^1$ is a complex domain and $H \subset C^1$ is an open half plane. Assume that for all the local operator $J(D)$ with constant coefficients the functions $J(D)f(z)$ are continuous up to the part of the boundary $(\partial H) \cap V$. Then, $f(z)$ admits an analytic continuation beyond this part.*

Remark. Though we have mentioned that the above Theorem is of local character, we don't know whether the following really local statement holds or not: Let f be a germ of hyperfunction at the origin. Assume that for any local operator J, $J(D)f$ is a germ of cotinuous function. Then f is a germ of real analytic function.

References

[1] Kaneko, A.: On the structure of hyperfunctions with compact supports (to appear in Proc. Japan Acad.).

[2] ——: On the representation of hyperfunctions by measures. Proceedings of Symposium of Hyperfunctions and Partial Differential Equations at RIMS, March, 1971, Kyoto Univ. (to appear) (in Japanese).

[3] ——: Representation of hyperfunctions by measures and some of its applications (in preparation).