168. *Freely Generable Classes of Structures*

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A class K of structures is said to be freely generable, if for any (non-empty) set E of generator symbols and any set Ω of defining relations, there exists a freely generated structure in K presented by E and Ω. The conditions for a class of algebras to be freely generable were studied in [1; § 8 in Chap. III] and [2]. Our main purpose of this note is to show a new necessary and sufficient condition for a class of structures to be freely generable.

A structure $\mathfrak{A}$ of the similarity type corresponding to a first order language L is simply called a structure for L. The domain of $\mathfrak{A}$ is denoted by $D[\mathfrak{A}]$. A formula Φ of L which contains at most some of $x_1, \cdots, x_n$ as free variables is denoted by $\Phi(x_1, \cdots, x_n)$ if the free variables $x_1, \cdots, x_n$ need to be indicated. Let $\Phi(x_1, \cdots, x_n)$ be any formula of L, and let $a_1, \cdots, a_n$ be elements in $D[\mathfrak{A}]$. Then we write $(\mathfrak{A}; a_1, \cdots, a_n) \models \Phi(x_1, \cdots, x_n)$, if $a_1, \cdots, a_n$ satisfy $\Phi(x_1, \cdots, x_n)$ in $\mathfrak{A}$ when the free variables $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively. An atomic formula of L means a formula of the form $t_1 = t_2$ or of the form $r(t_1, \cdots, t_m)$, where r is an m-ary relation symbol of L and $t_1, \cdots, t_m$ are terms of L. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for a first order language L. A mapping h of $D[\mathfrak{A}]$ onto (or into) $D[\mathfrak{B}]$ is called an L-homomorphism of $\mathfrak{A}$ onto (or into) $\mathfrak{B}$, if for any atomic formula $\Theta(x_1, \cdots, x_n)$ of L and for any elements $a_1, \cdots, a_n$ in $D[\mathfrak{A}]$, $(\mathfrak{A}; a_1, \cdots, a_n) \models \Theta(x_1, \cdots, x_n)$ implies $(\mathfrak{B}; h(a_1), \cdots, h(a_n)) \models \Theta(x_1, \cdots, x_n)$. An L-homomorphism h of $\mathfrak{A}$ onto $\mathfrak{B}$ is called an L-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if the mapping h is one to one and the inverse mapping h^{-1} is also an L-homomorphism. Let E be a set of constant symbols (i.e. nullary operation symbols) not belonging to L. Then, a new first order language can be obtained from L by adjoining all the constant symbols $e \in E$, which is denoted by $L(E)$. If $L(E)$ contains at least one constant symbol, then E is said to be L-generative. Now let $\mathfrak{A}$ be a structure for L, and ψ a mapping of E into $D[\mathfrak{A}]$. Then $\mathfrak{A}$ can be expanded to a structure for $L(E)$, by considering $\psi(e)$ as interpretations of e in $\mathfrak{A}$, and the expanded structure is denoted by $\mathfrak{A}(\psi)$.

Let K be a class of structures for L. Let E be a set of constant symbols not belonging to L, and Ω a set of atomic sentences (i.e. atomic formulas without free variables) of $L(E)$. Now let $\mathfrak{A}$ be a structure for L, and ψ a mapping of E into $D[\mathfrak{A}]$. The pair $(\mathfrak{A}, \psi)$ is called a *K-model of Ω with the set E of generator symbols*, if $\mathfrak{A}$ is in K and generated by $\psi(E)$ and the expanded structure $\mathfrak{A}(\psi)$ is a model of Ω. We denote by $[E, \Omega; K]$ the class of all K-models of Ω with the set E of generator symbols. A K-model $(\mathfrak{F}, \varphi)$ of Ω with the set E of generator symbols is said to be *freely generated*, if for any $(\mathfrak{A}, \psi) \in [E, \Omega; K]$, there exists an $L(E)$-homomorphism of $\mathfrak{F}(\varphi)$ onto $\mathfrak{A}(\psi)$, i.e. there exists an L-homomorphism of $\mathfrak{F}$ onto $\mathfrak{A}$ that maps $\varphi(e)$ to $\psi(e)$ for each $e \in E$. We denote by $F[E, \Omega; K]$ the class of all freely generated K-models of Ω with the set E of generator symbols. Note that if $(\mathfrak{F}, \varphi)$ and $(\mathfrak{F}', \varphi')$ are in $F[E, \Omega; K]$ then $\mathfrak{F}(\varphi)$ and $\mathfrak{F}'(\varphi')$ are $L(E)$-isomorphic. Now let K be a class of structures for L. K is said to be *freely generable*, if for any L-generative set E of constant symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, $F[E, \Omega; K]$ is not empty. K is said to be *conditionally freely generable*, if for any L-generative set E of constant symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, $[E, \Omega; K] \neq \emptyset$ implies $F[E, \Omega; K] \neq \emptyset$, where $\emptyset$ denotes the empty set.

Let K be a class of structures for L. We denote by $\boldsymbol{P}_s(K)$ the class of all subdirect products of non-empty families of structures in K, and by $\boldsymbol{P}_s^*(K)$ the class of all subdirect products of empty or non-empty families of structures in K, where the subdirect product of the empty family of structures for L means the one-element structure $\mathfrak{E}_L$ whose domain consists of only the empty set $\emptyset$ and in which every atomic formula of L is valid. Moreover we denote by $\boldsymbol{I}(K)$ the class of all L-isomorphic copies of structures in K. A class K of structures for L is said to be abstract if $\boldsymbol{I}(K) \subseteq K$.

Theorem 1. *Let K be a class of structures for a first order language L. Then, a necessary and sufficient condition for K to be conditionally freely generable is that $\boldsymbol{P}_s(K) \subseteq \boldsymbol{I}(K)$.*

Proof of necessity. Assume that K is conditionally freely generable. Let $(\mathfrak{A}_i \mid i \in I)$ be any non-empty family of structures in K, and let $\mathfrak{A}$ be any subdirect product of all $\mathfrak{A}_i, i \in I$. Then, it is sufficient to prove that $\mathfrak{A}$ is in $\boldsymbol{I}(K)$. Now let $E = \{e_a \mid a \in D[\mathfrak{A}]\}$ be a set of constant symbols not belonging to L, and Ω the set of all atomic sentences of $L(E)$ which are valid in $\mathfrak{A}(\psi)$, where ψ is the mapping of E onto $D[\mathfrak{A}]$ such that $\psi(e_a) = a$ for all $e_a \in E$. Moreover let ψ_i be the mapping of E onto $D[\mathfrak{A}_i]$ which maps e_a to $a(i)$, where $a(i)$ denotes the i-th component of a. Then it is clear that $(\mathfrak{A}_i, \psi_i)$ is in $[E, \Omega; K]$. Hence by the assumption, we have that $F[E, \Omega; K] \neq \emptyset$, i.e. there exists a freely generated K-model $(\mathfrak{F}, \varphi)$ belonging to $F[E, \Omega; K]$. Moreover there exists an $L(E)$-homomorphism h of $\mathfrak{A}(\psi)$ onto $\mathfrak{F}(\varphi)$, because every atomic sentence of $L(E)$ valid in $\mathfrak{A}(\psi)$ is also valid in $\mathfrak{F}(\varphi)$. On the other hand, for each $i \in \boldsymbol{I}$, there exists an $L(E)$-homomorphism of $\mathfrak{F}(\varphi)$ onto $\mathfrak{A}_i(\psi_i)$, because $(\mathfrak{F}, \varphi) \in F[E, \Omega ; K]$ and $(\mathfrak{A}_i, \psi_i) \in [E, \Omega ; K]$. Hence any atomic sentence Θ of $L(E)$ valid in $\mathfrak{F}(\varphi)$ is valid in every $\mathfrak{A}_i(\psi_i), i \in \boldsymbol{I}$. Hence the Θ is valid in $\mathfrak{A}(\psi)$. Therefore the $L(E)$-homomorphism h of $\mathfrak{A}(\psi)$ onto $\mathfrak{F}(\varphi)$ is an $L(E)$-isomorphism. Hence $\mathfrak{A}$ is L-isomorphic to $\mathfrak{F}$, and hence $\mathfrak{A}$ is in $\boldsymbol{I}(K)$. Therefore we have $\boldsymbol{P}_s(K) \subseteq \boldsymbol{I}(K)$.

Proof of sufficiency. Assume that $\boldsymbol{P}_s(K) \subseteq \boldsymbol{I}(K)$. Let E be any L-generative set of constant symbols not belonging to L, and let Ω be a set of atomic sentences of $L(E)$ such that $[E, \Omega ; K] \neq \emptyset$. We shall below prove that $F[E, \Omega ; K] \neq \emptyset$. It is easy to see that there exists a non-empty subset $\{(\mathfrak{A}_i, \psi_i) \mid i \in I\}$ of $[E, \Omega ; K]$ such that for any $(\mathfrak{B}, \psi) \in [E, \Omega ; K]$, some $\mathfrak{A}_i(\psi_i)$ is $L(E)$-isomorphic to $\mathfrak{B}(\psi)$. Now let $\mathfrak{A}$ be the direct product of all $\mathfrak{A}_i, i \in \boldsymbol{I}$, and let φ be the mapping of E into $D[\mathfrak{A}]$ such that for each $e \in E$ and for each $i \in \boldsymbol{I}$, the i-th component of $\varphi(e)$ is $\psi_i(e)$. Let $\mathfrak{F}$ be a substructure of $\mathfrak{A}$ generated by $\varphi(E)$. Then $\mathfrak{F}$ is a subdirect product of $\mathfrak{A}_i, i \in \boldsymbol{I}$, and hence by the assumption, $\mathfrak{F}$ is in $\boldsymbol{I}(K)$. And it is easy to see that $(\mathfrak{F}, \varphi)$ is in $[E, \Omega ; \boldsymbol{I}(K)]$. Moreover we have that for any $(\mathfrak{B}, \psi) \in [E, \Omega ; \boldsymbol{I}(K)]$, there exists an $L(E)$-homomorphism of $\mathfrak{F}(\varphi)$ onto $\mathfrak{B}(\psi)$, because some $\mathfrak{A}_i(\psi_i)$ and $\mathfrak{B}(\psi)$ are $L(E)$-isomorphic, and $\mathfrak{F}(\varphi)$ is clearly a subdirect product of all $\mathfrak{A}_i(\psi_i), i \in \boldsymbol{I}$. Therefore $(\mathfrak{F}, \varphi)$ is a freely generated $\boldsymbol{I}(K)$-model of Ω with the set E of generator symbols. Hence $F[E, \Omega ; \boldsymbol{I}(K)] \neq \emptyset$, and hence $F[E, \Omega ; K] \neq \emptyset$. Therefore we have that K is conditionally freely generable.

Theorem 2. *Let K be a class of structures for a first order language L. Then, a necessary and sufficient condition for K to be freely generable is that $\boldsymbol{P}_s^*(K) \subseteq \boldsymbol{I}(K)$.*

Proof. Suppose that K is freely generable. Then $\boldsymbol{P}_s(K) \subseteq \boldsymbol{I}(K)$ follows immediately from Theorem 1. Moreover it is easy to see that K contains a one-element structure for L in which every atomic formula of L is valid. Hence $\boldsymbol{I}(K)$ contains the one-element structure $\mathfrak{E}_L$ whose domain is $\{\emptyset\}$ and in which every atomic formula of L is valid. Hence we have $\boldsymbol{P}_s^*(K) \subseteq \boldsymbol{I}(K)$. Conversely, assume that $\boldsymbol{P}_s^*(K) \subseteq \boldsymbol{I}(K)$. Then K contains a one-element structure for L in which every atomic formula of L is valid. Hence $[E, \Omega ; K] \neq \emptyset$ holds for any L-generative set E of constant symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$. Hence by Theorem 1, K is freely generable.

As immediate consequences of Theorems 1 and 2, we have the following two theorems:

Theorem 3. *Let K be an abstract class of structures which are of the same type. Then*

(1) *K is conditionally freely generable if and only if $\boldsymbol{P}_s(K) \subseteq K$;*

(2) *K is freely generable if and only if $\boldsymbol{P}_s^*(K) \subseteq K$.*

Theorem 4. *Let K be a class of structures which are of the same type. Then*

(1) $\boldsymbol{I}(\boldsymbol{P}_s(K))$ *is the smallest conditionally freely generable abstract class containing the class K;*

(2) $\boldsymbol{I}(\boldsymbol{P}_s^*(K))$ *is the smallest freely generable abstract class containing the class K.*

References

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[2] T. Fujiwara: Note on free algebraic systems. Proc. Japan Acad., **32**, 662–664 (1956).